THÈSE

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La longue marche à travers le quart de plan

Sous la direction de **Mireille BOUSQUET-MÉLOU** et **Charlotte HARDOUIN**

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Chapitre 1

Introduction

On commence par présenter la combinatoire énumérative en général, ses outils et quelques techniques. On présente ensuite les chemins dans le quadrant, qui sont l'objet d'étude de la thèse.

1.1 Baguenaudage

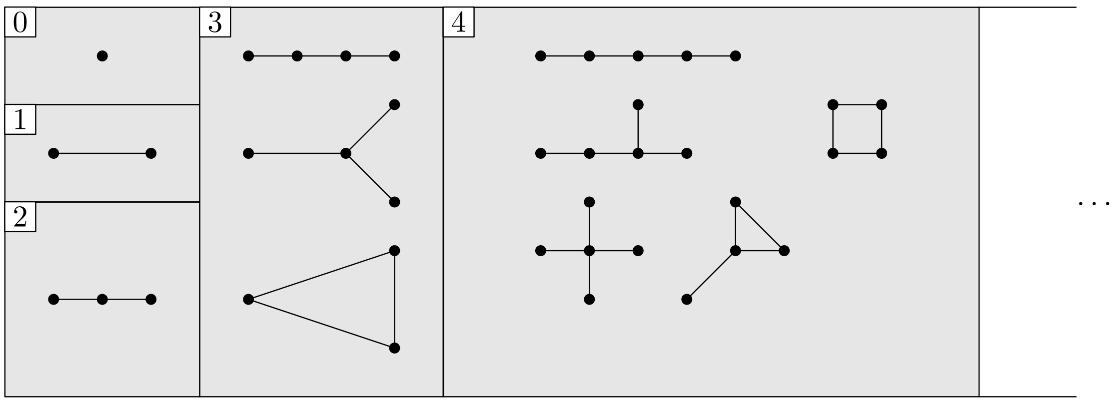
1.1.1 Dénombrer

La combinatoire énumérative vise à dénombrer des familles d'objets discrets : arbres, cartes combinatoires, mots, permutations, chemins, etc. Étant donnée une telle famille d'objets, on veut déterminer le nombre d'objets qui sont d'une certaine taille n. Par exemple, on peut compter les graphes simples connexes selon leur nombre d'arêtes (figure 1.1.1), les arbres binaires plans selon le nombre de nœuds internes, les mots binaires selon leur nombre de lettres évitant un certain motif, etc. En plus de son intérêt propre, cette discipline est en interconnexion avec beaucoup d'autres domaines des mathématiques, et s'avère également déterminante en informatique, notamment pour analyser la complexité d'algorithmes.

Une première approche pour le dénombrement consiste en la recherche de *bijections* explicites entre les objets que l'on souhaite dénombrer et une famille d'objets que l'on sait dénombrer. Elle est souvent liée à une compréhension profonde de la structure des objets étudiés, et donne lieu par exemple à des algorithmes de génération aléatoire, des algorithmes efficaces manipulant ces objets, des codes, etc.

Cette première approche est souvent difficile. Aussi, lorsque l'on souhaite surtout obtenir des résultats numériques sur de tels objets, on emploie une seconde approche, plus systématique, qui fait usage de la notion de *série génératrice*.

La série génératrice d'une famille d'objets $\mathcal{E}$ équipée d'une fonction de taille $|\cdot|$ (dite *classe combinatoire* [FS09]) est la série formelle $E(t) = \sum_{e\in\mathcal{E}} t^{|e|}$, où $|e|$ désigne la taille de l'objet e (par exemple le nombre d'arêtes pour un graphe, le nombre de lettres pour un mot, etc.). Il est nécessaire que le nombre d'objets d'une certaine taille fixée soit fini. Le coefficient de t^n dans $E(t)$ correspond alors au nombre d'objets de taille n dans



$$G(t) = 1 + t + t^2 + 3t^3 + 5t^4 + \dots$$

FIGURE 1.1.1 – Les graphes simples connexes comptés selon leur nombre d'arêtes.

la classe $\mathcal{E}$.

Pour obtenir des informations sur $E(t)$, une technique particulièrement fructueuse consiste à chercher une description inductive de la famille d'objets considérés, qui se traduit en *équation fonctionnelle* sur la série génératrice. Dans bien des cas, cette équation peut être résolue, donnant une forme explicite pour la série génératrice $E(t)$. Le cas échéant, des informations partielles peuvent souvent être déduites de l'équation fonctionnelle. On peut notamment obtenir des résultats asymptotiques sur les coefficients de la série $E(t)$ par des méthodes d'analyse complexe [FS09], ou encore construire des algorithmes efficaces de calcul des coefficients [Bos+17]. La nature des équations en jeu permet aussi de quantifier la complexité des objets étudiés. Une fois ce travail effectué, cette approche basée sur des équations fonctionnelles permet de faciliter l'approche bijective, en permettant de « deviner » que deux familles d'objets sont en bijection et donner des indices sur sa recherche.

Des chemins de fortune

Dans la suite de cette introduction, on illustre ces différentes méthodes de manière plus précise via l'étude d'une famille d'objets « jouet », celle des *préfixes de Dyck*, qui est assez simple.

Définition 1.1.1. Un *chemin* dans $\mathbb{Z}$ de longueur n est une suite finie de *pas* $(w_1, \dots, w_n) \in \mathbb{Z}^n$. Étant donnés deux chemins, $w = (w_1, \dots, w_n)$ et $w' = (w'_1, \dots, w'_m)$, leur concaténation est définie comme $w \cdot w' = (w_1, \dots, w_n, w'_1, \dots, w'_m)$.

Un *méandre* est un chemin w de longueur n de point de départ 0 et qui reste au dessus de 0. On veut donc que $\sum_{i=1}^{k} w_i \geq 0$ pour tout $k \leq n$. Une *excursion* est un méandre qui termine en 0. On veut donc que $\sum_{i=1}^{n} w_i = 0$.

Lorsque les pas w_i appartiennent à $\{-1, 1\}$, les méandres et excursions sont respectivement appelés *préfixes de Dyck* et *chemins de Dyck*.

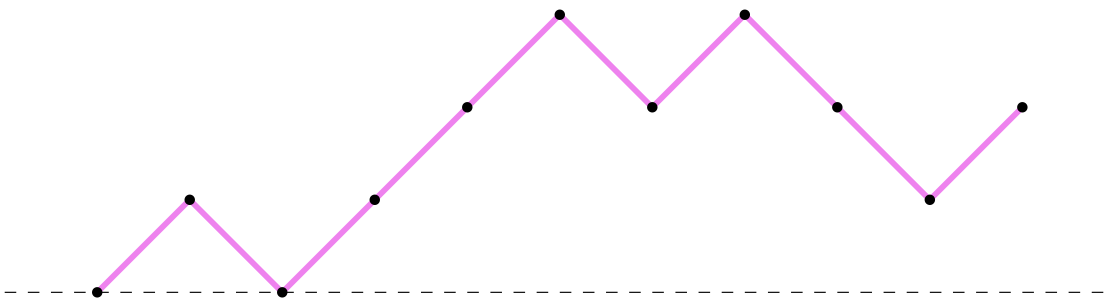


FIGURE 1.1.2 – Un préfixe de Dyck de longueur 10. La dimension horizontale représente le temps.

1.1.2 Bijections

Une première approche du dénombrement est la recherche d'une bijection entre la famille d'objets considérée avec un famille d'objets plus simple, dont le dénombrement est plus facile. Souvent, ces bijections relient également des propriétés structurelles de ces familles d'objets (par exemple, la bijection classique entre les arbres binaires plans comptés par nœuds internes et les chemins de Dyck comptés par nombre de pas relie le nombre de retour en zéro avec la profondeur de la branche la plus à droite). Cela a par exemple des applications à des algorithmes de génération aléatoire. Par exemple, on dispose d'un algorithme de génération aléatoire pour les arbres binaires plans (l'algorithme de Rémy [Rém85]), qui est donc directement applicable pour générer aléatoirement des chemins de Dyck.

On présente maintenant un exemple classique d'argument bijectif qui s'applique au dénombrement des chemins de Łukasiewicz. Un chemin de Łukasiewicz de longueur n est un chemin $(w_1, \dots, w_n)$ dont les pas sont dans $\{-1, 0, \dots\}$, partant de 0, et tel que $\sum_{i=1}^{k} w_i \geq 0$ pour tout $k < n$, avec $\sum_{i=1}^{n} w_i = -1$.

Étant donné un n-uplet $w = (w_1, \dots, w_n)$ on définit sa *rotation* comme le n-uplet

$$r(w) \stackrel{\text{def}}{=} (w_2, \dots, w_n, w_1)$$

et pour tout $i \in \{0, \dots, n\}$ la somme

$$s_i(w) \stackrel{\text{def}}{=} \sum_{j=1}^{i} w_j,$$

qui correspond à l'altitude du chemin w au i-ème pas.

Lemme 1.1.2 (Un lemme cyclique). *Si $w = (w_1, \dots, w_n) \in \{-1, 0, \dots\}^n$ est un chemin terminant en -1, alors il existe un unique $k \in \{0, \dots, n-1\}$ tel que $r^k(w)$ est un chemin de Łukasiewicz.*

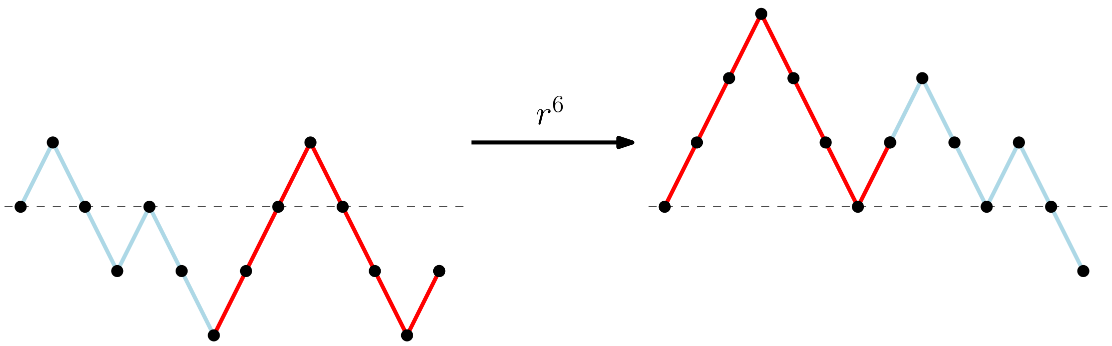


FIGURE 1.1.3 – On fait commencer le chemin à la première occurrence du record minimal

Démonstration. Soit $h = \min\{s_i(w) : i \in \{1, \ldots, n\}\}$ la hauteur minimale atteinte par le chemin, et $k = \min\{k \in \{1, \ldots, n\} : s_k(w) = h\}$ l'indice du pas de w qui réalise cette hauteur en premier (voir la Figure 1.1.3). En particulier, si $i \geq k$, on a $s_i(w) \geq s_k(w)$ et si $i < k$, on a $s_i(w) > s_k(w)$.

Considérons $i, j \in \{0, \ldots, n-1\}$ avec $i \neq j$. Si $i < j$, alors

$$\begin{aligned} s_{j-i}(r^i(w)) &= w_{i+1} + \cdots + w_j \\ &= s_j(w) - s_i(w). \end{aligned}$$

D'autre part, si $i > j$, alors

$$\begin{aligned} s_{n-i+j}(r^i(w)) &= w_{i+1} + \cdots + w_n + w_1 + \cdots + w_j \\ &= s_n(w) - (w_{j+1} + \cdots + w_i) \\ &= -1 + s_j(w) - s_i(w). \end{aligned}$$

On déduit de ces deux calculs avec $i = k$ que $r^k(w)$ est un chemin de Łucasiewicz. D'autre part, on déduit de ces deux calculs avec $j = k$ que seul ce k convient. □

Via le lemme cyclique, on en déduit entre autres une formule close du nombre de chemins de Dyck.

Corollaire 1.1.3. *Le nombre de chemins de Dyck de longueur* $2n$ *vaut*

$$\frac{1}{2n+1}\binom{2n+1}{n}.$$

Démonstration. On note

$$\begin{aligned} W_n &= \{(w_1, \ldots, w_n) \in \{-1, 1\}^n : s_n(w) = -1\} \\ D_n &= \{(w_1, \ldots, w_n) \in \{-1, 1\}^n : w \text{ est un chemin de Dyck}\}. \end{aligned}$$

Considérons l'application

$$\begin{aligned} \phi : \{0 \ldots 2n\} \times D_{2n} &\to W_{2n+1} \\ (k, w) &\mapsto r^k(w_1, \ldots, w_{2n}, -1). \end{aligned}$$

Le lemme cyclique 1.1.2 dit exactement que ϕ est une bijection. Elle est surjective : étant donné $w' \in W_{2n+1}$, le lemme cyclique donne un unique $k \in \{0, \dots, 2n\}$ tel que si $w \cdot z = r^{2n-k}(w')$ avec $z\{-1, 1\}$, le $2n$-uplet w est un préfixe de Dyck. Puisque $w \cdot z$ termine en -1, cela impose que $z < 0$. Puisque $z \in \{-1, 1\}$, on a nécessairement $z = -1$, et que w est un chemin de Dyck. Ainsi, $w' = \phi(k, w)$. Elle est injective : si $\phi(w) = \phi(w')$ et $k \geq k'$, alors $r^{k-k'}(w \cdot (-1)) = w' \cdot (-1)$. Puisque w' et w sont des préfixes de Dyck, le lemme cyclique impose que $k = k'$, et donc que $w = w'$. Aussi, on calcule le nombre de chemins de Dyck de longueur $2n$ comme

$$\#D_{2n} = \frac{\#W_{2n+1}}{2n+1} = \frac{1}{2n+1}\binom{2n+1}{n} \qquad \square$$

Il faut noter que la version présentée ici du lemme cyclique est restreinte. On peut en effet démontrer plus généralement que tout chemin utilisant les pas $\{-1, 0, \dots\}$ commençant en 0 et terminant en $-h-1$ est conjugué à une concaténation de $h+1$ chemins de Łukasiewicz, et ce de $h+1$ manières différentes (voir [Sta24, sec. 5.3] pour un développement). Via cet argument, on peut démontrer que le nombre de préfixes de Dyck de longueur $2n+h$ terminant à une hauteur h vaut

$$\frac{h+1}{2n+h+1}\binom{2n+h+1}{n}.$$

1.1.3 Séries génératrices et méthode symbolique

Bien que la construction de bijections soit une manière particulièrement élégante de dénombrer des objets, cette méthode demeure assez difficile à mettre en place. Elle nécessite en effet de saisir avec profondeur la structure des objets à dénombrer. Aussi, lorsque l'on souhaite avant tout des résultats numériques (quitte à ensuite trouver une bijection à partir de l'intuition conférée par ces résultats), on emploie la méthode de la « Generatingfunctionology » [Wil06].

Depuis Euler, on sait en effet que les séries génératrices sont un outil extrêmement puissant en combinatoire, et ce par le lien intime qu'il existe entre les constructions inductives d'objets (union disjointe, produit cartésien, etc) et les opérations algébriques sur les séries génératrices correspondantes (addition, multiplication, etc). Par suite, on obtient aisément des équations fonctionnelles sur les séries génératrices, que l'on peut alors soit résoudre explicitement, ou dont on peut du moins extraire des informations partielles sur la série génératrice solution.

On peut notamment trouver des relations de récurrence sur les coefficients, ou encore, et c'est l'objet de la *combinatoire analytique*, obtenir des estimations fines de leur comportement asymptotique. En effet, on peut dans de nombreux cas voir la série formelle comme une fonction d'une variable complexe, analytique en 0. L'asymptotique des coefficients se lit alors sur la position et la nature des singularités de la fonction au bord de son disque de convergence. Pour une très large classe de fonctions, l'asymptotique se détermine alors automatiquement. Pour d'autres types de séries vérifiant certaines équations (équations algébriques, équations différentielles), il est aussi possible d'obtenir des résultats asymptotiques sur les coefficients [FS09].

Séries formelles

On commence par introduire les séries formelles, comme généralisation des polynômes.

Définition 1.1.4. Soit R un anneau commutatif, et t une indéterminée. L'ensemble des *séries formelles* sur R, noté $R[\![t]\!]$, est défini comme l'ensemble des suites $(f_i)_{i\in\mathbb{N}} \in R^{\mathbb{N}}$, notées

$$F(t) = \sum_{n\geq 0} f_n t^n.$$

L'ensemble $R[\![t]\!]$ a une structure d'anneau commutatif, où la somme et le produit de $F(t) = \sum_{n\geq 0} f_n t^n$ et $G(t) = \sum_{n\geq 0} g_n t^n$ sont définis comme

$$F(t) + G(t) \stackrel{\text{def}}{=} \sum_{n\geq 0} (f_n + g_n) t^n$$

$$F(t) \cdot G(t) \stackrel{\text{def}}{=} \sum_{n\geq 0} \left(\sum_{i=0}^{n} f_i\, g_{n-i} \right) t^n.$$

L'anneau des *polynômes* sur R en l'indéterminée t, noté $R[t]$, est naturellement un sous-anneau de $R[\![t]\!]$ correspondant aux suites nulles à partir d'un certain rang. Les notations définies ci-dessous s'y appliqueront donc.

On note $R(\!(t)\!)$ l'anneau des séries de Laurent sur R, défini comme la localisation de $R[\![t]\!]$ en t, soit les séries de la forme $H(t)/t^m$ pour un certain $m > 0$ et $H(t)$ une série formelle. On les note également

$$F(t) = \sum_{n\geq -m} f_n t^n$$

Étant données $F(t)$ et $G(t)$ deux séries de Laurent en t, on note $F(t) = G(t) + O(t^n)$ si $F(t) - G(t) = t^n H(t)$ pour $H(t)$ une série formelle. On note $[t^n]F(t) = f_n$ le n-ième coefficient de $F(t)$.

Si $F(t) = f_{-m} t^{-m} + O(t^{-m+1})$ est une série de Laurent avec $f_{-m} \neq 0$, on définit sa *valuation* $v(F) = -m$. L'application $v : R(\!(t)\!) \to \mathbb{Z}$ vérifie $v(FG) = v(F) + v(G)$ et $v(F+G) \geq \min\{v(F), v(G)\}$.

Si t et t' sont deux indéterminées sur R, les anneaux $(R[\![t]\!])\,[\![t']\!]$ et $(R[\![t']\!])\,[\![t]\!]$ sont isomorphes. On note alors $R[\![t, t']\!]$ cet anneau l'anneau des séries formelles à deux variables. Par suite, si $t_1, \ldots, t_n$ sont des indéterminées, on notera $R[\![t_1, \ldots, t_n]\!]$ l'anneau des séries formelles sur les variables $t_1, \ldots, t_n$. Si $F(t_1, \ldots, t_n) \in R[\![t_1, \ldots, t_n]\!]$ est une série formelle, on peut la voir comme une série formelle de $(R[\![t_1, \ldots, t_{n-1}]\!])\,[\![t_n]\!]$. On définit alors $[t_n^i]F(t_1, \ldots, t_n)$ comme le coefficient en t_n^i de cette série en tant que série en t_n. En général, si $m = t_1^{d_1} \ldots t_n^{d_n}$ est un monôme, on définira inductivement

$$[t_1^{d_1} \ldots t_n^{d_n}]F(t_1, \ldots, t_n) = [t_n^{d_{n-1}}]([t_1^{d_1} \ldots t_{n-1}^{d_n}]F(t_1, \ldots, t_n)).$$

Proposition 1.1.5 (Section 3.1 de [Bos+17])**.** *Soit $(A_k(t))_k$ une suite de séries formelles telles que $A_k(t) = O(t^{v_k})$ avec $v_k \to \infty$. Alors il existe une unique série formelle notée $\sum_{k\geq 0} A_k(t)$ caractérisée de la manière suivante. Pour tout $n \geq 0$, il existe un $k_0 \geq 0$ tel que pour tout $k \geq k_0$, on a*

$$\left(\sum_{i\geq 0} A_i(t)\right) = \sum_{i=0}^{k} A_i(t) + O(t^n).$$

Exemple 1.1.6.

1. Si $A(t)$ est une série formelle telle que $A(t) = O(t)$, et $B(t) = \sum_{n\geq 0} b_n t^n$ est une série formelle, alors la suite de séries formelles $(A(t)^k)_k$ vérifie $A(t)^k = O(t^k)$, ce qui assure que la somme $\sum_{k\geq 0} b_k A(t)^k$ définit bien une série formelle, correspondant à la *composition* $B(A(t))$.
2. Si $A(t) = O(t)$, on définit son *pseudo-inverse* $\sum_{n\geq 0} A(t)^n$, qui est l'inverse multiplicatif de la série $1 - A(t)$. Il vient que toute série formelle $A(t) = f_0 + O(t)$ avec f_0 inversible dans R est inversible dans $R[\![t]\!]$.

Exemple 1.1.7 (Point fixe et définition inductive)**.** Soit $F(x,t) \in \mathbb{Q}[\![x,t]\!]$ une série formelle. On considère l'équation

$$X = tF(X,t) \tag{1.1.1}$$

d'inconnue X. Cette équation, dite de *point fixe* admet une unique solution $x(t) \in \mathbb{Q}[\![t]\!]$.

En effet, pour une série formelle $y(t) = O(t)$, la composition $tF(y(t),t)$ est bien définie. Cela permet de définir une application $\phi : t\mathbb{Q}[\![x,t]\!] \longmapsto t\mathbb{Q}[\![x,t]\!]$ définie par $\phi(y(t)) = tF(y(t),t)$. Si $x(t)$ et $y(t)$ sont dans $O(t)$, on a par la formule de Taylor [Bos+17, sec. 3.1] que

$$\phi(y(t)) = \phi(x(t)) + (y(t) - x(t)) \cdot O(t).$$

Ainsi, en définissant la suite de polynômes $A_n(t) \overset{\text{def}}{=} \phi^n(0)$ (ϕ^n étant l'itération de la composition), on a par l'identité ci-dessus et par récurrence que si $m \geq n$, alors $A_m(t) = A_n(t) + O(t^{n+1})$. Par la proposition 1.1.5, on a que $x(t) \overset{\text{def}}{=} \sum_{n\geq 0}(A_{n+1}(t) - A_n(t))$ définit bien une série formelle, qui par construction est l'unique solution de l'équation (1.1.1). En effet, pour tout n, on a $x(t) = A_n(t) + O(t^{n+1})$, donc par la formule de Taylor on a

$$\phi(x(t)) = \phi(A_n(t)) + O(t^{n+1}) = A_{n+1}(t) + O(t^{n+1}) = x(t) + O(t^{n+1}),$$

ce qui permet de conclure.

Par exemple, étant donnée $F(t) \in \mathbb{Q}[\![t]\!]$ une série formelle de la forme $F(t) = t + t^2V(t)$ avec $V(t)$ une série formelle, on peut définir la série formelle $G(t) = \frac{1}{1+tV(t)} \in \mathbb{Q}[\![t]\!]$ (comme un pseudo-inverse). Par la discussion précédente, l'équation

$$X = tG(X)$$

a une unique solution $x(t) \in \mathbb{Q}[\![t]\!]$, qui vérifie par construction $F(x(t)) = t$. La série $x(t)$ est donc l'inverse compositionnel à droite de $F(t)$. ∎

Les équations de point fixe interviennent très souvent dans un contexte combinatoire pour des objets définis inductivement, via la méthode symbolique que l'on présente juste après.

Classes combinatoires

On introduit maintenant le concept de *classe combinatoire*. Si cette notion n'est pas indispensable pour faire de la combinatoire, elle donne un cadre formel pour assurer un dictionnaire entre des objets issus de la combinatoire et leurs séries formelles associées (dites *séries génératrices*). La présentation élémentaire ci-dessous provient essentiellement du livre *Analytic Combinatorics* de Flajolet et Sedgewick [FS09], mais d'autres définitions plus générales existent, via par exemple la théorie des espèces combinatoires, qui permet notamment de dénombrer des objets à automorphisme près (voir l'ouvrage de référence [BLL98]).

Définition 1.1.8. Une *classe combinatoire* $\mathcal{A}$ est la donnée d'une paire $(\mathcal{A}, |\cdot|)$ où $\mathcal{A}$ est un ensemble et $|\cdot| : \mathcal{A} \to \mathbb{N}$ définit une partition de $\mathcal{A}$ en ensembles finis $\{x \in \mathcal{A} : |x| = n\}$. On note $\mathcal{A} \simeq \mathcal{B}$ s'il existe une bijection $\sigma : \mathcal{A} \to \mathcal{B}$ telle que pour tout a dans $\mathcal{A}$, $|\sigma(a)| = |a|$.

Définition 1.1.9. Étant données deux classes combinatoires, $\mathcal{A}$ et $\mathcal{B}$, on a (entre autres) les constructions suivantes :

- *Atome neutre* : $\mathit{1} = (\{*\}, (|*| = 0))$
- *Atome unité* : $\mathcal{T} = (\{*\}, (|*| = 1))$
- *Union disjointe* : $\mathcal{A} + \mathcal{B} = (\mathcal{A} \uplus \mathcal{B}, \mathbf{1}_{\mathcal{A}} \cdot |\cdot|_{\mathcal{A}} + \mathbf{1}_{\mathcal{B}} \cdot |\cdot|_{\mathcal{B}})$
- *Produit cartésien* : $\mathcal{A} \times \mathcal{B} = (\mathcal{A} \times \mathcal{B}, (|(x,y)| = |x|_{\mathcal{A}} + |y|_{\mathcal{B}}))$
- *Séquence* : $\mathcal{S}eq(\mathcal{A}) = 1 + \mathcal{A} + \mathcal{A}^2 + \dots$. Cette construction ne fonctionne que si $\mathcal{A}$ n'a pas d'objet de taille nulle (auquel cas, il y aurait une infinité d'objets de taille 0 dans $\mathcal{S}eq(\mathcal{A})$ avec cette définition).

Exemple 1.1.10. On liste ici diverses classes combinatoires associées à des objets usuels.

- *Mots binaires* On définit $\mathcal{S}$ la classe des mots sur l'alphabet $\{0, 1\}$, la taille d'un mot étant le nombre de lettres qui le composent. Un mot est soit vide, soit est la concaténation de la lettre 0 ou de la lettre 1 avec un mot quelconque. Si on note $\mathcal{T}_0$ l'atome correspondant à la lettre 0, et $\mathcal{T}_1$ l'atome correspondant à la lettre 1, cela se traduit en
$$\mathcal{S} \simeq 1 + \mathcal{T}_0 \times \mathcal{S} + \mathcal{T}_1 \times \mathcal{S}.$$
Plus directement, un mot est une suite finie de lettres, donc on a
$$\mathcal{S} \simeq \mathcal{S}eq(\mathcal{T}_0 + \mathcal{T}_1).$$
- *Arbres binaires plans* On définit $\mathcal{A}$ la classe des arbres binaires plans, la taille d'un arbre étant son nombre de nœuds internes. Un arbre binaire plan est soit composé d'un seul nœud (qui n'est pas interne), soit constitué d'un nœud interne et de deux sous-arbres. Si on note $\mathcal{T}_{\text{int}}$ l'atome correspondant à un nœud interne, cela se traduit en
$$\mathcal{A} \simeq 1 + \mathcal{T}_{\text{int}} \times \mathcal{A} \times \mathcal{A}.$$

■

De la même manière que dans l'exemple 1.1.7 où dans certaines conditions il est possible de définir une série formelle via une équation de point fixe, on peut définir sous certaines conditions une classe combinatoire comme unique solution (à isomorphisme de classe combinatoire près) d'une équation de point fixe de classes combinatoires, via la notion de *construction admissible* (voir [FS09, sec. I.2]). Aussi, les isomorphismes de classes combinatoires constatés dans l'exemple 1.1.10 pour des classes définies au préalable, auraient suffi à spécifier les classes $\mathcal{S}$ et $\mathcal{A}$.

Définition 1.1.11 (Série génératrice ordinaire)**.** Soit $\mathcal{A}$ une classe combinatoire. On définit $A(t)$ la *série génératrice ordinaire* de $\mathcal{A}$ comme la série formelle

$$A(t) \stackrel{\text{def}}{=} \sum_{a \in \mathcal{A}} t^{|a|},$$

de sorte que $\#\{a \in \mathcal{A} : |a| = n\} = [t^n]A(t)$.

Cette définition trouve sa pertinence dans la correspondance suivante entre les opérations sur les classes combinatoires et leurs séries génératrices ordinaires respectives.

Proposition 1.1.12. *Il y a la correspondance suivante entre ces constructions de classes combinatoires et leurs séries génératrices respectives.*

Classe combinatoire	*Série génératrice*
1	1
$\mathcal{T}$	t
$\mathcal{A} + \mathcal{B}$	$A(t) + B(t)$
$\mathcal{A} \times \mathcal{B}$	$A(t) \cdot B(t)$
$\mathcal{S}eq(\mathcal{A})$	$(1 - A(t))^{-1}$

Dans la majorité des cas, on se contentera de décrire directement l'équation fonctionnelle vérifiée par la série génératrice d'une classe combinatoire définie inductivement, étant donné cette transparence vis-à-vis des constructions algébriques de classes combinatoires. On note que deux classes combinatoires équivalentes ont la même série génératrice ordinaire.

Exemple 1.1.13 (Composition d'entiers)**.** Un entier naturel non nul s'identifie à son développement en unaire, dont le nombre de chiffres correspond à sa taille (on a $|n| = n$). On a donc que la classe des entiers naturels non-nuls est donnée par

$$\mathcal{E} \simeq \mathcal{T} \times \mathcal{S}eq\,(\mathcal{T})$$

Une *composition d'entiers* se voit comme une suite d'entiers naturels non nuls (dont la taille est la somme des entiers qui la compose), on a donc

$$\mathcal{C} \simeq \mathcal{S}eq\,(\mathcal{E})\,.$$

Ainsi, la série génératrice des entiers naturels non nuls vaut

$$E(t) = \frac{t}{1-t},$$

et la série génératrice des compositions d'entiers vaut

$$C(t) = \frac{1}{1 - \frac{t}{1-t}} = \frac{1}{1-2t} - \frac{t}{1-2t}.$$

On retrouve alors qu'il y a 2^{n-1} compositions d'entiers sommant à n lorsque $n \geq 1$. ■

On regarde maintenant plus de paramètres associés aux objets dénombrés.

Définition 1.1.14. Soit $\mathcal{A}$ une classe combinatoire. Une statistique χ est une application de $\mathcal{A}$ dans $\mathbb{N}$. On regarde $\chi = (\chi_1, \ldots, \chi_d)$ une famille finie de statistiques sur $\mathcal{A}$. Étant donnée une telle paire $< \mathcal{A}, \chi >$, et $x1$, ..., x_n des variables, on définit sa série génératrice

$$A(t) \stackrel{\text{def}}{=} \sum_{a \in \mathcal{A}} x_1^{\chi_1(a)} \ldots x_d^{\chi_d(a)} t^{|a|}.$$

Sans plus rentrer dans les détails formels (voir [FS09, sec. III.3]), pour la plupart des statistiques considérées et naturelles, le dictionnaire de la méthode symbolique vaut toujours, c'est-à-dire que pour des classes combinatoires munies de statistiques supplémentaires, les opérations d'union disjointe et produit cartésien donnent lieu à des sommes et produits des séries génératrices correspondantes.

Exemple 1.1.15. Soit $\mathcal{A}$ la classe des mots binaires sans le motif 000, où l'on compte le nombre de fois k où 0 apparaît. Alors sa série génératrice

$$A(x,t) = \sum_{n \geq 0} a_{n,k} x^k t^n$$

où $a_{n,k} = \#\{w \in \mathcal{A} \;:\; w \text{ a } n \text{ lettres et } 0 \text{ apparaît } k \text{ fois}\}$ se détermine inductivement sur le nombre de lettres. On considère les classes $\mathcal{A}_i$ des mots de $\mathcal{A}$ dont le plus grand suffixe constitué de 0 est de longueur i, et on note $\mathcal{T}_d$ la classe de la lettre $d \in \{0,1\}$. On note $A_i(x,t)$ la série génératrice correspondante à $\mathcal{A}_i$. Par induction sur la longueur du mot, ces classes combinatoires vérifient les relations suivantes :

$$\begin{aligned}
\mathcal{A}_0 &\simeq \varepsilon + \mathcal{A}_0 \times \mathcal{T}_1, \\
\mathcal{A}_1 &\simeq \mathcal{A}_0 \times \mathcal{T}_0, \\
\mathcal{A}_2 &\simeq \mathcal{A}_1 \times \mathcal{T}_0, \\
\mathcal{A} &\simeq \mathcal{A}_0 + \mathcal{A}_1 + \mathcal{A}_2.
\end{aligned}$$

Ces équivalences se traduisent en un système linéaire sur les séries génératrices

$$\begin{aligned} A_0(x,t) &= 1 + A(x,t)\cdot t, \\ A_1(x,t) &= A_0(x,t)\cdot(xt), \\ A_2(x,t) &= A_1(x,t)\cdot(xt), \\ A(x,t) &= A_0(x,t) + A_1(x,t) + A_2(x,t). \end{aligned}$$

En résolvant ce système, on obtient pour $A(x,t)$ l'expression

$$A(x,t) = \frac{1 + xt + x^2t^2}{1 - t - xt^2 - x^2t^3}.$$

■

Cet appareil formel étant posé, on revient désormais à notre exemple fil rouge des préfixes de Dyck.

Exemple 1.1.16 (Préfixes de Dyck II)**.** On considère $\mathcal{A}$ la classe des préfixes de Dyck, la taille d'un préfixe de Dyck c étant son nombre de pas. Étant donné un préfixe de Dyck c, on considère deux statistiques : n_x le nombre de fois où c atteint 0 (les *contacts*), et i l'altitude finale de c. Lorsque l'on considère la statistique des contacts avec l'axe, on parle de chemins *à bords interactifs*. On définit pour tout i la classe $\mathcal{D}_i$ des chemins commençant et terminant à l'altitude i, et restant au dessus de i à tout instant. La classe des chemins de Dyck est donc $\mathcal{D}_0$. On note

$$D_i(a,t) = \sum_{c\in\mathcal{D}_i} a^{n_x} t^{|c|}$$

et

$$A(x,a,t) = \sum_{c\in\mathcal{A}} x^i a^{n_x} t^{|c|}.$$

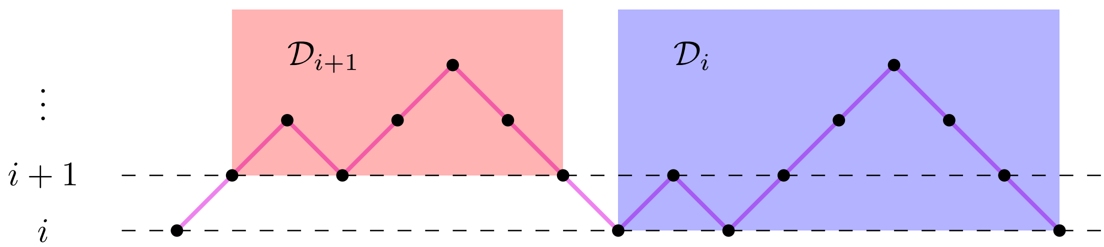


FIGURE 1.1.4 – Décomposition des chemins de Dyck

Dans un premier temps, on a naturellement (Figure 1.1.4) la décomposition

$$\mathcal{D}_i \simeq 1 + \mathcal{T}_\uparrow \times \mathcal{D}_{i+1} \times \mathcal{T}_\downarrow \times \mathcal{D}_i.$$

Elle se traduit en deux identités de séries génératrices, selon que i est nul ou non. On a d'une part que $D_i(a,t) = D_1(a,t)$ pour tout $i \geq 1$, étant donné qu'il n'y a pas de contact avec les axes. Ces séries sont indépendantes de a, donc on les écrit $D_i(t)$. Aussi, on a

$$D_1(t) = 1 + (xt) \cdot D_1(t) \cdot (t/x) \cdot D_1(t). \tag{1.1.2}$$

D'autre part, pour $D_0(a,t)$, le premier pas descendant de 1 à 0 non trivial ajoute un contact avec l'axe des abscisses, ce qui donne l'équation

$$D_0(a,t) = 1 + (xt) \cdot D_1(t) \cdot (at/x) \cdot D_0(a,t). \tag{1.1.3}$$

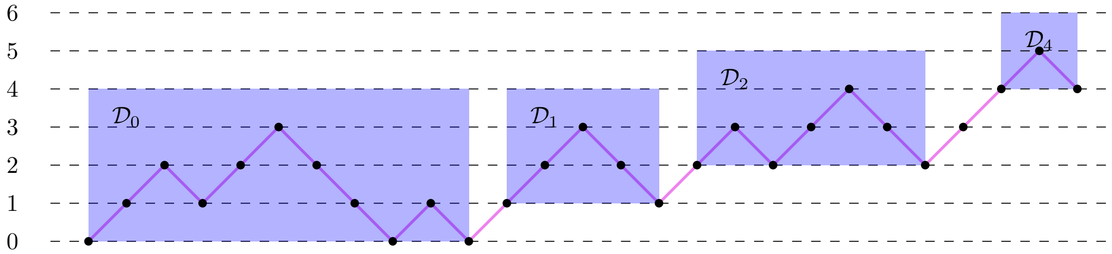


FIGURE 1.1.5 – Décomposition des préfixes de Dyck

Dans un second temps, on peut décomposer $\mathcal{A}_i$ la classe des préfixes de Dyck terminant à l'altitude i comme dans la figure 1.1.5, ce qui donne l'équivalence

$$\mathcal{A}_i \simeq \mathcal{D}_0 \times \mathcal{T}_\uparrow \times \mathcal{D}_1 \times \cdots \times \mathcal{T}_\uparrow \times \mathcal{D}_i.$$

Puisque par définition $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1 + \ldots$, on obtient alors la décomposition

$$\begin{aligned}\mathcal{A} \simeq\ & \mathcal{D}_0 \\ & + \mathcal{D}_0 \times \mathcal{T}_\uparrow \times \mathcal{D}_1 \\ & + \mathcal{D}_0 \times \mathcal{T}_\uparrow \times \mathcal{D}_1 \times \mathcal{T}_\uparrow \times \mathcal{D}_2 \\ & + \ldots\end{aligned}$$

On obtient alors finalement l'équation

$$A(x,a,t) = D_0(a,t) \cdot \sum_{i \geq 0} (xtD_1(t))^i = \frac{D_0(a,t)}{1 - xtD_1(t)}. \tag{1.1.4}$$

En combinant (1.1.2), (1.1.3) et (1.1.4), on peut donc déterminer explicitement la série génératrice $A(x,a,t)$ des préfixes de Dyck avec statistiques de la hauteur finale et du nombre de contacts

$$\begin{aligned} A(x,a,t) &= \frac{1}{(1 - at^2 D_1(t))\,(1 - xtD_1(t))}, \\ \text{avec } D_1(t) &= \frac{1 - \sqrt{1 - 4t^2}}{2t^2}. \end{aligned} \tag{1.1.5}$$

■

1.1.4 Équations aux variables catalytiques

On a vu dans la section précédente que l'on pouvait se dispenser de la recherche d'une bijection astucieuse pour dénombrer des préfixes de Dyck, via une décomposition inductive. Cette décomposition donnant lieu à une équation algébrique assez simple sur les séries génératrices, elle donne une expression explicite de la série génératrice, que l'on peut ensuite manipuler afin d'extraire des informations sur les objets dénombrés. En outre, la flexibilité de cette approche permet d'incorporer au dénombrement diverses statistiques associées aux chemins dénombrés.

Néanmoins, cette décomposition inductive bien connue des préfixes de Dyck repose sur le fait qu'il n'y a qu'un seul type de pas descendant. La *Generatingfunctionology* permet toutefois de se dispenser de trouver une telle décomposition ingénieuse pour obtenir des informations sur la série génératrice, via des variables *catalytiques* (le terme est dû à Doron Zeilberger).

Exemple 1.1.17 (Préfixes de Dyck III)**.** Soit $\mathcal{A}$ la classe des préfixes de Dyck telle que définie dans l'exemple 1.1.16. On réintroduit aussi $\mathcal{A}_i$ la classe des préfixes de Dyck qui terminent à l'altitude i, et $\mathcal{A}_{>i}$ la classe des préfixes de Dyck qui terminent à une altitude strictement plus grande que i.

On définit encore la série formelle de $\mathcal{A}$ comme $A(x,a,t) \overset{\text{def}}{=} \sum_{c\in\mathcal{A}} x^i a^{n_x} t^n$, avec i l'altitude finale, n_x le nombre de contacts, et n le nombre de pas de c. Puisque $R[x][\![t]\!] \subset R[\![x,t]\!]$, on peut définir $A_0(a,t) = [x^0]A(x,a,t)$ et $xA_1(a,t) = x[x^1]A(x,a,t)$, qui par définition sont les séries génératrices associées aux classes $\mathcal{A}_0$ et $\mathcal{A}_1$.

Pour former une équation aux variables catalytiques sur $A(x,a,t)$, on va simplement décrire inductivement les chemins selon leur nombre de pas, en maintenant la statistique de la hauteur terminale d'un chemin, qui est codée via la variable *catalytique* x. Selon cette description, un préfixe de Dyck est soit

— un préfixe de Dyck trivial, sans pas,

— un préfixe de Dyck auquel on a ajouté l'un des deux pas, $+1$ ou -1. L'ajout du pas 1 est possible à la fin de tout chemin, tandis que celui du pas -1 ne peut se faire que si le chemin termine strictement au dessus de l'axe.

Cette description inductive se traduit en l'équation

$$\mathcal{A} \simeq 1 + \mathcal{T}_\uparrow \times \mathcal{A} + \mathcal{T}_\downarrow \times \mathcal{A}_{>1} + \mathcal{T}_\downarrow \times \mathcal{A}_{=1}.$$

On obtient alors une équation fonctionnelle sur la série génératrice $A(x,a,t)$, qui est

$$\begin{aligned} A(x,a,t) &= 1 + (tx)\cdot A(x,a,t) \\ &\quad + \frac{t}{x}\cdot\big(A(x,a,t) - xA_1(a,t) - A_0(a,t)\big) + \frac{at}{x}\cdot(xA_1(a,t)). \end{aligned} \tag{1.1.6}$$

Cette équation donne une relation de dépendance polynomiale entre la série $A(x,a,t)$, et des séries qui ne dépendent pas de la variable x, à savoir $A_1(a,t)$ et $A_0(a,t)$, elles aussi inconnues, que l'on appelle des *sections*. On regroupe donc les termes correspondant à $A(x,a,t)$ à gauche de l'équation, et on multiplie le tout par x pour obtenir

$$\begin{aligned} K(x,t)A(x,a,t) &= x + (a-1)xtA_1(a,t) - tA_0(a,t) \\ \text{où } K(x,t) &= x - t(1+x^2). \end{aligned} \tag{1.1.7}$$

Cette équation est qualifiée d'équation *à noyau*, le nom provenant de celui du polynôme $K(x,t)$ qui est en facteur de $A(x,t)$ dans le côté gauche de l'équation. Le côté droit de l'équation a la particularité que les fonctions inconnues qui subsistent sont indépendantes de la variable x. Cette configuration permet alors à elle seule de déterminer les séries $A_1(t)$, $A_0(t)$ et $A(x,t)$, via la *méthode du noyau* (voir [Knu69, p.536-537]). Elle consiste à trouver des relations algébriques entre les séries inconnues du membre de droite en évaluant l'équation sur des séries formelles $x(t)$ telles que $K(x(t),t) = 0$ (d'où le qualificatif de noyau).

Étudions donc l'équation $K(x(t),t) = 0$. On trouve deux racines, dont

$$x_0(t) = \frac{1-\sqrt{1-4t^2}}{2t} = t + t^2 + 2t^3 + O(t^4)$$

qui est une série formelle en t. Puisque $A(x,a,t) = \sum_{i\geq 0} x^i A_i(a,t)$ est une série formelle à coefficients polynomiaux en x, la composition $A(x_0(t),a,t)$ est bien définie. Ainsi, on peut évaluer l'équation en $x = x_0(t)$, ce qui donne une première relation

$$0 = x_0(t) + (a-1)x_0(t)tA_1(a,t) - tA_0(a,t). \tag{1.1.8}$$

D'autre part, en utilisant le fait que $A(x,a,t) = A_0(a,t) + xA_1(a,t) + O(x^2)$, l'injection de cette identité dans l'équation donne en considérant le coefficient en x l'autre relation

$$A_0(a,t) - tA_1(a,t) = 1 + (a-1)tA_1(a,t) \tag{1.1.9}$$

en considérant le coefficient de x. Cette identité se voit également combinatoirement : un chemin terminant en 0 est soit trivial, soit un chemin qui termine en 1 suivi d'un pas descendant donnant lieu à un contact. Cependant il faut noter qu'on l'a obtenue uniquement via l'équation fonctionnelle.

Via ces deux équations, on retrouve les expressions de $A_0(a,t)$ et $A_1(a,t)$ comme

$$\begin{aligned} A_0(a,t) &= \frac{x_0(t)}{x_0(t) + at - ax_0(t)} \\ A_1(a,t) &= \frac{x_0(t) - t}{t(ax_0(t) - at - x_0(t))}. \end{aligned}$$

En réinjectant ces expressions dans l'équation (1.1.7), on retrouve la même expression de $A(x,a,t)$ qu'en (1.1.5). ∎

1.1.5 Analyse de singularités

On a vu dans les sections précédentes diverses manières d'obtenir les expressions de séries génératrices pour des objets combinatoires, notamment la série génératrice $A(x,a,t)$ des préfixes de Dyck à bords interactifs. Il s'agit à présent d'exploiter cette expression, et on choisit ici de présenter l'*analyse de singularités*, qui permet d'obtenir assez facilement des développements asymptotiques pour les coefficients d'une série formelle. La connaissance de cette asymptotique permet notamment d'établir des comportements limites pour des objets de grande taille. On se contentera de rappeler ici ce résultat* :

Proposition 1.1.18 (Corollaire VI.1 dans [FS09]). *Soit $\sum_{n\geq 0} a_n t^n \in \mathbb{C}[[t]]$ une série entière qui admet un prolongement analytique $A(t)$ sur l'ouvert $D(0,r)\setminus[\rho,\infty[$ pour un $r>\rho>0$ (on rappelle que $D(0,r)$ est le disque ouvert centré en 0 de rayon r). Alors si $A(t)\sim(1-t/\rho)^\alpha$ au voisinage de 1 pour un certain $\alpha<0$, on a $a_n\sim\rho^{-n}\frac{n^{-\alpha-1}}{\Gamma(-\alpha)}$. Pour une série admettant un nombre fini de telles singularités, il suffit de sommer leurs contributions en le rayon de convergence afin d'avoir l'asymptotique globale des coefficients.*

On définit une probabilité sur les préfixes de Dyck de longueur n comme suit. Étant donné un nombre réel positif a, si c est un préfixe de Dyck de longueur n avec n_x contacts avec l'axe, on définit son poids comme a^{n_x}. On définit alors une probabilité sur les préfixes de longueur n de sorte que le chemin c soit pondéré par a^{n_x}, i.e. la probabilité de c vaut

$$\mathbb{P}_n(c)=\frac{a^{n_x}}{[t^n]A(1,a,t)}.$$

On propose d'étudier le comportement limite de trois quantités définies sur ces marches. On note X_n la variable aléatoire sur les marches de taille n qui correspond à l'altitude finale, et Z_n le nombre de contacts avec 0. On cherche alors à déterminer :

1. La probabilité limite de terminer en 0, définie par

$$p_0 \stackrel{\text{def}}{=} \lim_n \mathbb{P}_n(X_n=0)=\lim_n \frac{[t^n]A(0,a,t)}{[t^n]A(1,a,t)}.$$

2. L'asymptotique de la hauteur moyenne où termine la marche, définie par

$$h(n) \stackrel{\text{def}}{=} \mathbb{E}(X_n)=\frac{[t^n](x\partial_x A)(1,a,t)}{[t^n]A(1,a,t)}.$$

3. L'asymptotique du nombre moyen de contacts avec l'axe, soit

$$c(n) \stackrel{\text{def}}{=} \mathbb{E}(Z_n)=\frac{[t^n](a\partial_a A)(1,a,t)}{[t^n]A(1,a,t)}.$$

*. Le théorème de transfert [FS09, Theorem VI.1] donne des conditions plus générales sur la fonction $A(t)$ pour relier son comportement asymptotique au bord du disque de convergence avec le comportement asymptotique des coefficients. La condition plus faible énoncée dans la proposition 1.1.18 suffit dans notre cas, où l'on étudie des branches de séries algébriques (définies en section 1.1.6).

	$a<2$	$a=2$	$a>2$
$[t^n]A(1,a,t)$	$\frac{\sqrt{2}}{\sqrt{\pi}(2-a)}n^{-1/2}\,2^n$	2^{n-1}	$\frac{(-2+a)(a-\sqrt{a-1}-1)}{2(a-1)(a-2\sqrt{a-1})}r^{-n}$
$[t^n]A(0,a,t)$	$\frac{a\sqrt{2}}{\sqrt{\pi}(2-a)^2}n^{-3/2}\,2^n$	$\sqrt{\frac{2}{\pi}}\,n^{-1/2}\,2^{n-1}$	$\frac{a-2}{2(a-1)}r^{-n}$
$[t^n](x\partial_x A)(1,a,t)$	$\frac{1}{2-a}2^n$	$\sqrt{\frac{2}{\pi}}\,n^{1/2}\,2^{n-1}$	$\frac{a-2}{2\sqrt{a-1}(\sqrt{a-1}-1)}r^{-n}$
$[t^n](a\partial_a A)(1,a,t)$	$\frac{a\sqrt{2}}{\sqrt{\pi}(2-a)^2}n^{-1/2}\,2^n$	$\sqrt{\frac{2}{\pi}}\,n^{1/2}\,2^{n-1}$	$\frac{(a-2)^2}{4(a-1)^{3/2}(\sqrt{a-1}-1)}n\,r^{-n}$

(a) Asymptotiques des séries intermédiaires

	$a<2$	$a=2$	$a>2$
p_0	0	0	$1-\frac{1}{\sqrt{a-1}}$
$h(n)$	$\sqrt{\frac{\pi}{2}}n^{1/2}$	$\sqrt{\frac{2}{\pi}}n^{1/2}$	$\frac{a-2}{2(a-1)}$
$c(n)$	$\frac{a}{2-a}$	$\sqrt{\frac{2}{\pi}}n^{1/2}$	$\frac{1-(a-1)^2}{1+(a-1)^{1/2}}\frac{n}{4}$

(b) Statistiques limites

FIGURE 1.1.6 – Comportements asymptotiques obtenues via l'étude de $A(x,a,t)$ (on note $r = \frac{\sqrt{a-1}}{a}$)

Puisque la série $A(x,a,t)$ satisfait une équation polynomiale de degré 2 à coefficients dans $\mathbb{Q}(x,a,t)$ (voir (1.1.5)), les séries $A(1,a,t)$, $A(0,a,t)$, $(x\partial_x A)(1,a,t)$ et $(a\partial_x A)(1,a,t)$ satisfont des équations de degré 2 à coefficients dans $\mathbb{Q}(a,t)$. De fait, en spécialisant a à un réel positif, la proposition 1.1.18 s'applique à ces séries, et on pourra déterminer le comportement asymptotique des quantités définies ci-dessus via l'analyse des singularités de ces séries.

Les calculs sont détaillés dans la feuille Maple relative à cette section. Pour chacune de ces fonctions, on suit les étapes suivantes :

— On détermine le polynôme minimal de la série concernée.

— Via [FS09, th. VII.7], on identifie une liste finie de candidats possibles pour être la singularité dominante (la plus proche de l'origine) de la fonction concernée. On effectue alors un développement asymptotique de la solution au voisinage de cette valeur via la méthode de Newton [FS09, Th. VII.1]. La singularité dominante est alors la plus petite valeur qui donne lieu à une singularité.

— On applique la proposition 1.1.18 à la fonction pour obtenir le tableau 1.1.6a, puis par quotient des estimations on en déduit le tableau 1.1.6b.

L'interprétation du tableau 1.1.6b permet de déterminer trois comportements possibles selon la valeur de a, illustrés en figure 1.1.7 :

— Pour $a<2$, on a que la hauteur moyenne finale $h(n)$ tend vers l'infini, et que le nombre de contacts $c(n)$ est borné. Aussi, on caractérise ce comportement de *répulsif* (la marche s'éloigne de l'axe).

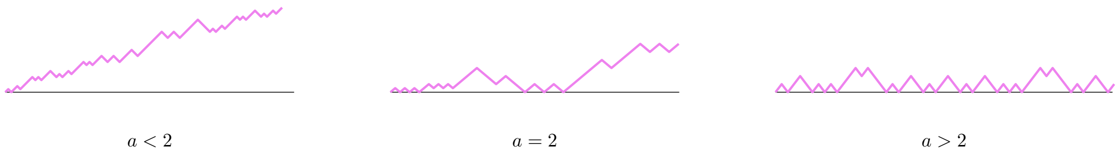


FIGURE 1.1.7 – Trois comportements limites

— Pour $a = 2$, on a à la fois que la hauteur moyenne finale $h(n)$ et que le nombre de contacts $c(n)$ tendent vers l'infini. On caractérise ce comportement de *critique*, au sens où il mélange les deux caractéristiques répulsive et attractive. La marche termine arbitrairement haut tout en maintenant un nombre élevé de contacts.

— Pour $a > 2$, on a que la hauteur moyenne $h(n)$ est finie, et que le nombre de contacts $c(n)$ est linéaire, on caractérise ce comportement d'*attractif* (la marche reste très proche de l'axe).

1.1.6 Nature des séries

Avant de passer à proprement à parler à la présentation des marches contraintes, on définit dans cette section une hiérarchie de séries formelles, dite *algébro-différentielle*, qui donne une mesure de la complexité des séries formelles en plusieurs variables, qui en retour traduit la complexité des objets dont elles sont les séries génératrices. Par souci de simplicité, on considère des séries formelles dans $\mathbb{C}[x,y][[t]]$, mais les notions s'étendent naturellement à plus d'indéterminées.

Séries rationnelles

Définition 1.1.19. Une série formelle $A(x,y,t) \in \mathbb{C}[x,y][[t]]$ est dite *rationnelle* si elle peut s'écrire $A(x,y,t) = \frac{P(x,y,t)}{Q(x,y,t)}$ pour $P(x,y,t)$ et $Q(x,y,t)$ dans des polynômes en x, y et t.

Les séries rationnelles ont été étudiées de manière intensive, et dans le cas d'une seule variable (série dans $\mathbb{C}(t)$), presque toutes les questions asymptotiques [FS09, th. IV.5.1] ou de calcul [Bos+17] à leur sujet sont résolues (il reste toutefois des problèmes difficiles, par exemple le problème de Skolem [Tao07]). Dans un contexte combinatoire, les séries rationnelles sont souvent associées à des séries génératrices de langages rationnels[†], de chemins dans un graphe fini, ou des chemins non-contraints. Les séries rationnelles forment un sous-anneau de l'anneau $\mathbb{C}[x,y][[t]]$.

†. On a notamment le théorème de représentation de Kleene-Schutzenberger qui donne l'équivalence entre le fait qu'une certaine série formelle associée à un langage soit N-rationnelle (ici, cela veut dire construite à partir de l'addition, du produit et du pseudo inverse $\frac{1}{1-z}$) et la reconnaissance de ce langage par un automate [Sak09, Theorem 3.5].

Séries algébriques

Une généralisation naturelle de la notion de série rationnelle est la notion de série algébrique. On rappelle brièvement ici que si k est un sous-corps de L, un élément α de L est algébrique sur k s'il existe un polynôme $P(Z) \in k[Z]$ tel que $P(\alpha) = 0$ (voir la définition 2.1.2) C'est donc tout naturellement que l'on définit les séries algébriques.

Définition 1.1.20. Une série formelle $A(x, y, t)$ est dite *algébrique* s'il existe un polynôme non-nul $P \in \mathbb{C}[F, x, y, t]$ tel que $P(A(x, y, t), x, y, t) = 0$.

Les séries algébriques sont typiquement associées à des langages décrits par des grammaires hors-contextes non-ambiguës ou des séries génératrices de chemins contraints dans un demi-plan. Elles forment un anneau.

Étant donné un polynôme à coefficients dans $\mathbb{C}(t)$, ses racines ont la particularité d'avoir une description explicite, via les *séries de Puiseux*.

Définition 1.1.21. Soit $\mathbb{K}$ un corps algébriquement clos de caractéristique nulle. On définit l'anneau des *séries fractionnaires* en t comme la limite inductive (grosso modo la réunion des anneaux)

$$\mathbb{K}^{frac}[\![t]\!] \stackrel{\text{def}}{=} \lim_{d\to\infty} \mathbb{K}[\![t^{1/d}]\!]$$

Puisque que $\mathbb{K}[\![t^{1/d}]\!]$ et $\mathbb{K}[\![t^{1/d'}]\!]$ sont dans $\mathbb{K}[\![t^{1/(dd')}]\!]$, on peut définir la somme et le produit de deux séries de Puiseux en les considérant comme des séries formelles en $t^{1/d}$ pour un d assez grand. Le corps des fractions de $\mathbb{K}^{frac}[\![t]\!]$ est appelé corps des *séries de Puiseux*, et est noté $\mathbb{K}^{frac}((t))$. Si

$$F(t) = a_{-m/d} t^{-m/d} + O(t^{(-m+1)/d})$$

avec $a_{-m/d} \neq 0$, sa valuation $v(F)$ vaut $-m/d$. La valuation des séries de Puiseux est donc à valeurs dans $\mathbb{Q}$.

Le théorème de Newton-Puiseux assure que le corps $\mathbb{K}^{frac}((t))$ est une clôture algébrique du corps des séries de Laurent $\mathbb{K}((t))$ [Sta24]. Cela signifie que pour $\mathbb{K}$ une clôture algébrique de $\overline{\mathbb{C}(x, y)}$ les séries algébriques $A(x, y, t)$ ont un développement explicite en série de Puiseux dans $\mathbb{K}^{frac}((t))$. En particulier, pour les séries algébriques dans $\mathbb{C}[\![t]\!]$, l'analyse de singularité est une question entièrement résolue (cf [FS09, par. VII.7]).

Séries différentiellement finies

L'extension naturelle des séries algébriques est l'incorporation d'équations différentielles linéaires. On commence par définir les séries différentiellement finies dans le cas des fonctions d'une seule variable.

Définition 1.1.22. Soit $A(t) \in k[\![t]\!]$ une série formelle sur un corps k. Alors $A(t)$ est dite t-D-finie sur k (ou différentiellement finie en la variable t) si $A(t)$ est solution d'une équation différentielle linéaire à coefficients dans $k[t]$, c'est-à-dire s'il existe $L_t \in k[t, \partial_t]$ non nul tel que $L_t(A(t)) = 0$. Lorsque la variable est évidente, on parle simplement de série différentiellement finie.

Cette définition permet de bonnes propriétés de clôture : les séries t-D-finies forment un anneau, cette classe est stable par intégration, et par composition avec des séries algébriques. En particulier, les séries algébriques sont D-finies. En outre, il y a équivalence entre le fait que la série soit D-finie et le fait que ses coefficients vérifient une relation de récurrence linéaire à coefficients polynomiaux, ce qui donne des algorithmes de calcul efficaces pour les calculer. En particulier, ces algorithmes sont appliqués aux séries algébriques (qui sont D-finies) afin de calculer rapidement leurs coefficients (la méthode de Newton est surtout utilisée pour déterminer les premiers coefficients, qui donnent l'initialisation de cette suite récurrente). Voir [Sta24, ch. 6] ou [Bos+17, Partie 3].

Lorsque l'on souhaite étendre cette définition à des séries de plusieurs variables, une manière de procéder est la suivante, due à [Lip89], qui permet justement à toutes les propriétés de clôture énoncées plus haut d'être préservées.

Définition 1.1.23. Une série formelle $A(x, y, t) \in \mathbb{C}[x, y][[t]]$ est dite *différentiellement finie* (abrégé D-finie) si elle est à la fois t-D-finie, x-D-finie (on la voit comme une série de $(\mathbb{C}[[y, t]])[[x]]$) et y-D-finie.

Séries différentiellement algébriques

La dernière généralisation considérée est la classe des séries *différentiellement algébriques*, où les équations différentielles présentes ne sont plus forcément linéaires. Tout comme dans le cas des séries différentiellement finies, on définit d'abord les séries différentiellement algébriques en une seule variable.

Définition 1.1.24. Une série formelle $A(t) \in k[[t]]$ est dite t-D-algébrique (différentiellement algébrique selon la variable t) lorsqu'elle est solution d'une équation différentielle polynomiale sur $k(t)$, c'est-à-dire s'il existe un polynôme P non nul à coefficients dans k tel que

$$P(t, A(t), \partial_t A(t), (\partial_t)^2 A(t), \dots) = 0.$$

Cette classe vérifie toutes les propriétés de clôture de la classe des séries différentiellement finies, en plus du fait qu'elle est stable par quotients et compositions. On définit alors une notion d'algébricité différentielle pour des séries à plusieurs variables de manière à ce que ces propriétés de clôture restent respectées.

Définition 1.1.25. Une série formelle $A(x, y, t)$ dans $\mathbb{C}[[x, y, t]]$ est dite *différentiellement algébrique* (abrégé D-algébrique) si elle est à la fois x-D-algébrique, y-D-algébrique et t-D-algébrique, c'est-à-dire s'il existe trois polynômes P_x, P_y et P_t tels que

$$P_x(x, y, t, A(x, y, t), \partial_x A(x, y, t), (\partial_x)^2 A(x, y, t), \dots) = 0,$$
$$P_y(x, y, t, A(x, y, t), \partial_y A(x, y, t), (\partial_y)^2 A(x, y, t), \dots) = 0,$$
$$P_t(x, y, t, A(x, y, t), \partial_t A(x, y, t), (\partial_t)^2 A(x, y, t), \dots) = 0.$$

1.2 Marches dans un cône

On présente maintenant l'objet d'étude de la thèse, soit l'énumération de chemins contraints dans un cône en deux dimensions.

On considère $\mathcal{S} \subset \mathbb{Z}^2 \setminus \{(0,0)\}$ un ensemble fini de directions, appelé *l'ensemble des pas* (ou *modèle*). Chacun de ces pas $s \in \mathcal{S}$ pourra éventuellement être muni d'un poids complexe d_s non nul. On appelle naturellement un ensemble de pas muni de poids un *modèle à poids* (*weighted model* en anglais). Notamment, les nombres d_s peuvent être choisis algébriquement indépendants sur $\mathbb{Q}$, ce qui permet de les traiter comme des variables. On se donne également un point de départ $s_0 \stackrel{\text{def}}{=} (i_0, j_0) \in \mathbb{Z}^2$, choisi par défaut comme étant $(0,0)$. On se donne enfin C un cône de $\mathbb{R}^2$, c'est-à-dire un sous-ensemble de $\mathbb{R}^2$ vérifiant pour (x,y) dans C que $\lambda \cdot (x,y)$ appartient à C pour tout $\lambda \geq 0$.

On cherche à dénombrer les chemins utilisant les pas de $\mathcal{S}$, en prenant en compte leurs poids, partant du point (i_0, j_0), et qui sont confinés dans C. Cela veut dire que pour le point de départ s_0, et $(w_1, \ldots, w_n) \in \mathcal{S}^n$, on veut pour tout $k \in \{0, \ldots, n\}$ que le point $s_0 + \sum_{i \leq k} w_i$ appartienne à C.

On est amené naturellement à vouloir décrire la série génératrice $W_C(x,y,t) \in \mathbb{C}[x, 1/x, y, 1/y][\![t]\!]$ de ces chemins définie par

$$W_C(x,y,t) \stackrel{\text{def}}{=} \sum_{w \text{ chemin}} \left(\prod_{s \in \mathcal{S}} d_s^{n_s} \right) x^i y^j t^n.$$

Dans la somme, la paire (i,j) désigne le point où termine le chemin w partant du point s_0, l'entier n_s désigne le nombre d'occurrence du pas s dans le chemin w, et n est le nombre total de pas utilisés par w. On abrégera souvent $W_C(x,y,t)$ par $W_C(x,y)$. On introduit le *polynôme des pas*

$$S(x,y) = \sum_{(i,j) \in \mathcal{S}} d_{i,j} x^i y^j \in \mathbb{C}[x, 1/x, y, 1/y].$$

1.2.1 Marches non-contraintes

On considère d'abord le cas où le cône C est trivial, c'est-à-dire égal à tout l'espace $\mathbb{R}^2$, ce qui revient à ne mettre aucune contrainte sur les chemins. Sans perte de généralité, on fait partir le chemin de $(0,0)$.

L'absence de contrainte permet la décomposition inductive suivante. Soit le chemin est trivial, soit il est obtenu en ajoutant un pas $s \in \mathcal{S}$ à la fin de n'importe quel chemin existant. L'équation fonctionnelle s'écrit alors

$$W_C(x,y) = 1 + \sum_{i,j} ([x^i y^j] S(x,y)) x^i y^j t W_C(x,y),$$

et on a donc

$$W_C(x,y) = \frac{1}{1 - tS(x,y)}.$$

La série génératrice est donc rationnelle. De fait, on peut assez simplement l'exploiter pour obtenir des informations sur les chemins. Par exemple, il est possible de décrire la série génératrice des marches qui terminent à un point donné (i, j), qui correspond à

$$[x^i y^j] W_C(x, y) = \sum_{n \geq 0} \left([x^i y^j] S(x, y)^n \right) t^n.$$

Cette extraction utilise le théorème des résidus, successivement pour obtenir $[y^j] W_C(x, y)$ (une série *algébrique*, car c'est une diagonale d'une série rationnelle, voir par exemple [BF02]), puis $[x^i] \left([y^j] W_C(x, y) \right)$ (une série *D-finie*, car c'est une diagonale d'une série algébrique, voir [Lip89]).

1.2.2 Marches contraintes dans un demi-plan

On considère le cas où le cône C est un demi-plan, défini par une équation $ax + by \geq 0$ pour des coefficients a et b réels. Cette fois-ci, tous les mots sur l'ensemble des pas ne produisent pas une marche licite, et le langage de ces mots n'est même plus automatiquement rationnel. En effet, s'il existe des cas triviaux (par exemple on n'aura pas grand peine à déterminer la série génératrice des chemins contraints dans $x \geq 0$ pour le modèle donné par $S(x, y) = x \ldots$), la série génératrice des chemins de Dyck est algébrique non rationnelle, donc le langage des mots qui codent un chemin de Dyck (le langage de Dyck) n'est pas reconnaissable par un automate fini.

Lorsque a et b sont quelconques, il y a peu d'espoir de donner une réponse systématique à cette question. Dans le cas où a/b est rationnel, on peut toujours via un reparamétrage rationnel se ramener à l'étude d'un modèle de chemins dans le demi-plan $y \geq 0$. Via l'emploi d'une variable catalytique, on va comme vu en 1.1.4 pouvoir former une équation fonctionnelle sur la série génératrice $H(y) \stackrel{\text{def}}{=} W_C(x, y, t)$. Ici, la contrainte ne porte que sur les ordonnées, donc l'équation catalytique formée à la manière de la section précédente est en la variable y. Puisque $\mathbb{C}[x, 1/x, y][[t]] \subset \mathbb{C}[x, 1/x][[t, y]]$, on peut définir

$$H_i(x, t) \stackrel{\text{def}}{=} [y^i] H(y) \in \mathbb{C}[x, 1/x][[t]]$$

la série génératrice des chemins terminant à l'altitude i, et

$$H_{<i}(y) \stackrel{\text{def}}{=} \sum_{i' < i} H_{i'}(x, t) y^{i'} \in (\mathbb{C}[x, 1/x][[t]])\, [y]$$

la série génératrice des chemins terminant strictement en dessous de l'altitude i. Cette dernière série est un polynôme en y, de degré strictement plus petit que i.

On peut alors écrire l'équation fonctionnelle comme suit, en décomposant inductivement les chemins selon le nombre de pas utilisés. Le cas de base correspond au chemin trivial, de point de départ (i_0, j_0). Pour le cas inductif, on se demande pour chaque pas de $\mathcal{S}$ dans quelle mesure on peut l'ajouter à un chemin existant. S'il avance en la coordonnée y, on peut l'ajouter à tout chemin existant. S'il recule de i, on peut l'ajouter à un chemin dès lors qu'il termine à une ordonnée plus grande que i.

Exemple 1.2.1. On regarde le cas où $\mathcal{S} = \{(1,-2),(0,-1),(-2,1)\}$ et où l'on part de 0.

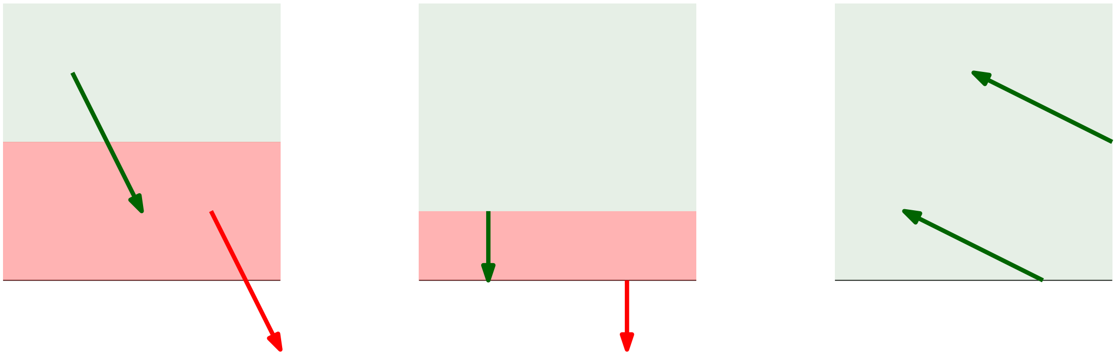


FIGURE 1.2.1 – Un pas peut être ajouté à tout chemin dès lors qu'il reste dans le demi-plan

L'équation fonctionnelle s'écrit

$$H(y) = 1 + tx^{-2}yH(y) + t/y(H(y) - H_0(y)) + tx/y^2(H(y) - H_1(y) - H_0(y)).$$

■

La décomposition inductive décrite précédemment se traduit dans le cas général en l'équation fonctionnelle

$$H(y) = x^{i_0}y^{j_0} + \sum_{i\geq 0} ty^i([y^i]S(x,y))H(y) + \sum_{i>0} ty^{-i}([y^{-i}]S(x,y))(H(y) - H_{<i}(y)). \tag{1.2.1}$$

En regroupant à gauche les termes en $H(y)$, et regroupant à droite les facteurs en $H_i(x,t)$, l'équation fonctionnelle se réécrit en

$$x^{-i_0}(1 - tS(x,y))H(y) = y^{j_0} - t\sum_{i=0}^{m} a_i(x,y)H_i(x,t),$$

où $-m$ est la valuation en y de $S(x,y)$ (la taille du plus grand pas qui recule en y), où les $a_i(x,y) \in \mathbb{C}[x,1/x,1/y]$ sont connus, et vérifient $\deg_y a_i(x,y) < 0$ pour tout i.

Théorème 1.2.2. *La série génératrice $H(y)$ des chemins partant de $(i_0,0)$ est algébrique.*

Démonstration. Comme dans l'exemple traité en Section 1.1.4, on emploie la méthode du noyau. Le développement suivant est dû à [BP00]. On multiplie l'équation par y^m pour m minimal tel que les coefficients de l'équation sont à coefficients polynomiaux ($-m$ est la valuation de $S(x,y)$ en y, correspondant donc aux pas de $\mathcal{S}$ qui reculent le plus), et on obtient alors l'équation

$$x^{-i_0}(y^m - ty^mS(x,y))H(y) = y^m + t\sum_i y^m a_i(x,y)H_i(x,t).$$

La proposition 6.1.8 de [Sta24] donne que le nombre de racines dans $\overline{\mathbb{C}(x)}^{frac}[\![t]\!]$ de l'équation $K(x,y) \stackrel{\text{def}}{=} y^m - ty^m S(x,y) = 0$ est m. De plus, il est facile de voir que les racines de ce polynôme sont distinctes, aussi on considère $y_1(x,t), \dots, y_m(x,t)$ les m racines distinctes dans $\mathbb{K}^{frac}[\![t]\!]$, et $y_{m+1}(x,t), \dots$ les autres racines en y de $K(x,y,t)$. En substituant $y_i(x,t)$ à y dans l'équation, on obtient m équations

$$0 = y_i(x,t)^m + t \sum_i y_i(x,t)^m a_i(x, y_i(x,t)) F_i(x).$$

Puisque $\deg_y a_i(x,y) < 0$, le polynôme unitaire $Y^m + t\sum_i Y^m a_i(x,Y) F_i(x,t)$ est de degré m en Y, de coefficient dominant Y^m. Les $y_i(x,t)$ en sont m racines distinctes, par conséquent il peut se factoriser comme

$$Y^m + t \sum_i y^m a_i(x,Y) F_i(x,t) = (Y - y_1(x,t)) \dots (Y - y_m(x,t)).$$

Ainsi, $H(y)$ est algébrique, et vaut

$$H(y) = \frac{(y - y_1(x)) \dots (y - y_m(x))}{K(x,y)}.$$

En particulier, la série $H_i(x,t)$ des chemins allant de $(i_0, 0)$ à une altitude l pour $l \in \mathbb{N}$ est algébrique, puisque l'expression ci-dessus est rationnelle en y. □

Il faut noter que l'argument précédent utilisait de manière cruciale le fait que le point de départ des chemins considérés était $y = 0$. Autrement, le membre de droite aurait eu un degré strictement plus grand que m, et la connaissance de seulement m de ses racines n'aurait pas suffi pour le factoriser. On peut toutefois réutiliser simplement le théorème 1.2.2 pour un point de départ général, et sans trop de calculs supplémentaires.

Corollaire 1.2.3. *La série génératrice $H(y)$ des chemins partant de l'altitude $k \geq 0$ est algébrique.*

Démonstration. Notons $H_{k \to l}(t)$ la série génératrice des chemins dans le demi-plan supérieur allant de k à l.

Montrons d'abord par induction sur $k \geq 0$ que $H_{k \to 0}(t)$ est algébrique. Le cas de base $k = 0$ a été traité précédemment. Pour le cas inductif, on utilise le fait qu'un chemin qui va de k à 0 peut se décomposer comme une suite de chemins demeurant dans le demi-plan supérieur, séparés par des pas descendants qui chacun réalisent le premier record à une altitude inférieure (figure 1.2.2). C'est la décomposition par premier passage à l'altitude inférieure.

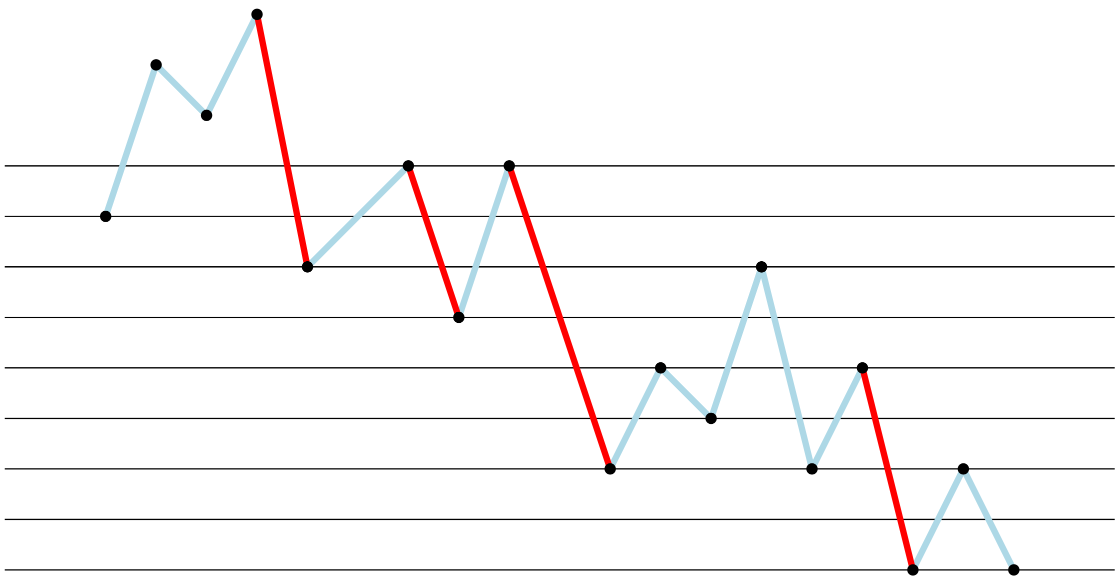


FIGURE 1.2.2 – Décomposition par premier passage inférieur

Aussi, la série génératrice $H_{k\to 0}(t)$ s'écrit

$$H_{k\to 0}(t) = \sum_{i,j \,/\, i-j<0} H_{0\to i}(t) \cdot (t([y^{-j}]S(x,y)) \cdot H_{i-j\to 0}(t).$$

Puisque l'ensemble des pas descendants est fini, cette somme est finie, ce qui montre l'algébricité de $H_{k\to 0}(t)$.

Pour le cas général, il suffit alors de décomposer les chemins qui vont de de k à l par leur altitude minimale $r \in \{0,\dots,k\}$. Un chemin de k à l d'altitude minimale r se décompose alors en chemin partant k restant strictement au dessus de r, puis un pas descendant vers r, puis un chemin restant au dessus de r allant vers l. On écrit ainsi

$$H_{k\to l}(t) = \sum_{r=0}^{k} \sum_{j>0} H_{k-r-j\to 0}(t) \cdot t([y^{-j}]S(x,y)) \cdot H_{0\to l-r}(t).$$

De même que précédemment, cette somme est finie, ce qui montre l'algébricité de $H_{k\to l}(t)$ pour tous $k,l \geq 0$. On conclut à l'aide de l'équation fonctionnelle (1.2.1) que $H(y)$ est algébrique, puisqu'elle s'exprime comme une une fraction rationnelle d'un nombre fini de $(H_{k\to l}(t))_l$. □

On obtient alors de même que précédemment que la série des chemins terminant à une ordonnée j donnée est algébrique sur $\mathbb{C}(x,t)$ (c'est l'un des coefficients de $H(y)$), et que par suite la série des chemins terminant à une coordonnée (i,j) donnée est D-finie comme diagonale d'une série algébrique [Lip88]. On note qu'il existe des preuves alternatives d'algébricité de toutes ces séries solutions d'une équation d'une variable catalytique, via le théorème 2.2.1.

1.2.3 Marches contraintes dans le quart de plan

Jusqu'ici, les séries $W_C(x,y)$ comptant les chemins sans contraintes sur leur point d'arrivée sont algébriques. Par extraction, la série des chemins terminant à une coordonnée spécifique du plan est au plus D-finie. Cela change lorsque l'on arrive au cas général, à savoir lorsque le cône est constitué de deux demi-droites sécantes (figure 1.2.3). De même que précédemment, on peut lorsque les coordonnées de ces droites sont rationnelles se ramener via des reparamétrages de l'ensemble des pas à deux situations : soit le cône est donné par l'équation $x \geq 0 \vee y \geq 0$ (appelé le trois-quarts de plan), soit le cône convexe est donné par l'équation $x \geq 0 \wedge y \geq 0$ (appelé le quart de plan). On ne parlera ici que du deuxième cas, le quart de plan.

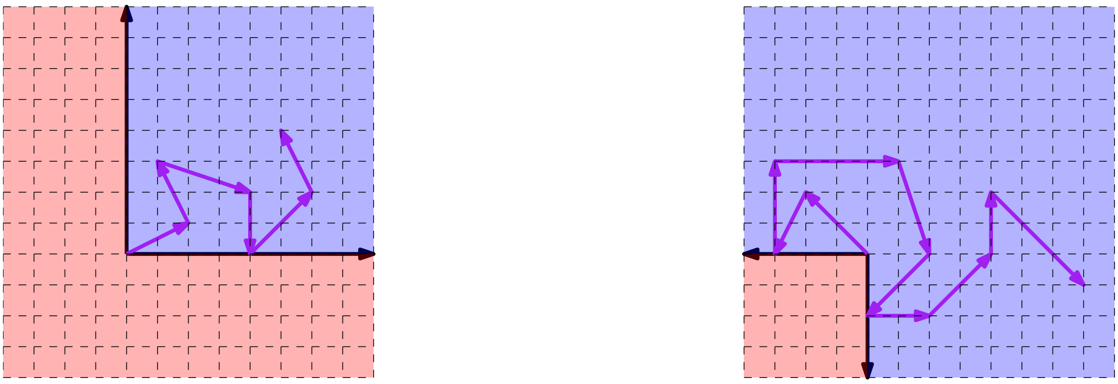


FIGURE 1.2.3 – Le quart de plan et le trois-quart de plan

Dans la situation de marches dans le quart de plan, il est aisé de former une équation fonctionnelle sur la série génératrice $Q(x,y) \stackrel{\text{def}}{=} W_C(x,y,t)$, au moyen des variables catalytiques x et y, qui vont coder les coordonnées où terminent les chemins considérés. On va décomposer inductivement les chemins, selon le nombre de pas utilisés. Le cas de base est le chemin trivial partant du point (i_0, j_0). Pour le cas inductif, on va comme précédemment considérer chacun des pas de $\mathcal{S}$, et déterminer à quelle condition sur le pas $(i,j) \in \mathcal{S}$ on peut l'ajouter à un chemin existant sans quitter le quart de plan.

1. Si le pas vérifie $i \geq 0$ et $j \geq 0$, on peut l'ajouter à n'importe quel chemin.
2. Si le pas vérifie $i \geq 0$ et $j < 0$, on peut l'ajouter à n'importe quel chemin qui termine hors du rectangle $[0,\infty] \times [0,-j[$.
3. De même, si le pas vérifie $i < 0$ et $j \geq 0$, on peut l'ajouter à n'importe quel chemin qui termine hors du rectangle $[0,-i[\times[0,\infty]$.
4. Enfin, si le pas vérifie $i < 0$ et $j < 0$, on peut l'ajouter aux chemins qui terminent hors du rectangle $[0,-i[\times[0,-j[$.

Étant donné que $Q(x,y) \in \mathbb{C}[x,y][[t]] \subset \mathbb{C}[[x,y,t]]$, on l'écrit de trois manières différentes, pour définir trois familles de séries génératrices :

$$Q(x,y) = \sum_{i,j\geq 0} Q_{i,j}(t)x^i y^j = \sum_{i\geq 0} Q_{i,\bullet}(y)x^i = \sum_{j\geq 0} Q_{\bullet,j}(x)y^i.$$

Il faut noter que les séries $Q_{i,j}(t)$, $Q_{i,\bullet}(y)$ et $Q_{j,\bullet}(x)$, que l'on appelle les *sections* peuvent s'exprimer comme

$$\begin{aligned} Q_{i,j}(t) &= \frac{(\partial_x^i \partial_y^j Q(x,y))(0,0)}{i!j!} \\ Q_{i,\bullet}(y) &= \frac{(\partial_x^i Q(x,y))(x=0)}{i!} \\ Q_{\bullet,j}(x) &= \frac{(\partial_y^j Q(x,y))(y=0)}{j!}. \end{aligned}$$

Il faut également noter qu'elles ne dépendent chacune que d'au plus une variable x et y. On introduit alors la série génératrice des chemins qui terminent dans un rectangle $[0,i[\times[0,j[$ pour tous $i \in \mathbb{N} \cup \{\infty\}$ et $j \in \mathbb{N} \cup \{\infty\}$,

$$Q_{<i,<j}(x,y) \stackrel{\text{def}}{=} \sum_{i'<i, j'<j} \left([x^{i'} y^{j'}] Q(x,y) \right) x^{i'} y^{j'}.$$

Les séries $Q_{<i,<j}(x,y)$ s'expriment comme des polynômes en x, y, et les séries $Q_{i,j}(t)$, $Q_{i,\bullet}(y)$ et $Q_{\bullet,j}(x)$.

Exemple 1.2.4. On considère le cas où $\mathcal{S} = \{(-1,-1),(1,3),(-2,0)\}$, le poids du pas (i,j) vaut $d_{i,j}$ et où on part de $(3,4)$.

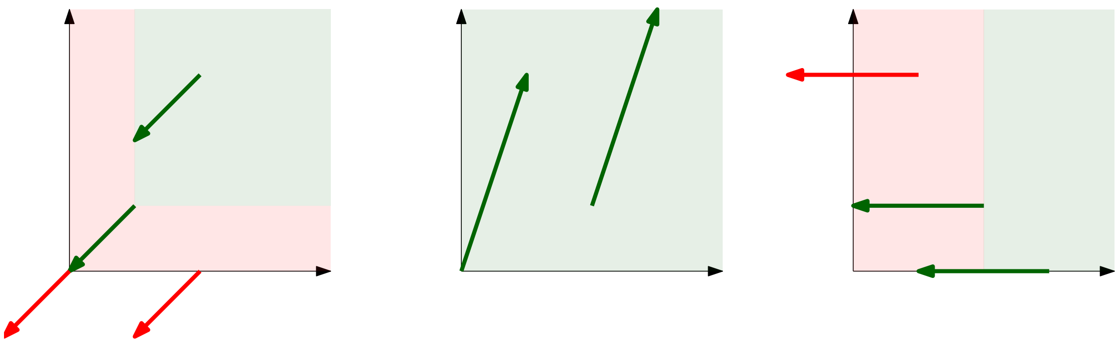


FIGURE 1.2.4 – Chaque pas de $\mathcal{S}$ peut être ajouté à un chemin existant pourvu que cet ajout ne quitte pas le quadrant.

Un chemin est soit trivial, soit construit en ajoutant un pas à l'une des marches terminant dans sa zone verte (figure 1.2.4). L'équation fonctionnelle s'écrit alors

$$\begin{aligned} Q(x,y) = x^3y^4 &+ td_{1,3}xy^3Q(x,y) + td_{-2,0}/x^2(Q(x,y) - Q_{0,\bullet}(y) - xQ_{1,\bullet}(y)) \\ &+ td_{-1,-1}/(xy)(Q(x,y) - Q_{0,\bullet}(y) - Q_{\bullet,0}(y) + Q_{0,0}(y)). \end{aligned}$$

■

L'équation fonctionnelle s'écrit de manière générale comme

$$\begin{aligned} Q(x,y) &= x^{i_0}y^{j_0} \\ &+ \sum_{i\geq 0, j\geq 0} tx^iy^j \left([x^iy^j]S(x,y)\right) Q(x,y) \\ &+ \sum_{i\geq 0, j>0} tx^iy^{-j} \left([x^iy^{-j}]S(x,y)\right) \left(Q(x,y) - Q_{<\infty,<j}(x,y)\right) \\ &+ \sum_{i>0, j\geq 0} tx^{-i}y^j \left([x^{-i}y^j]S(x,y)\right) \left(Q(x,y) - Q_{<i,<\infty}(x,y)\right) \\ &+ \sum_{i>0, j>0} tx^{-i}y^{-j} \left([x^{-i}y^{-j}]S(x,y)\right) \left(Q(x,y) - Q_{<i,<j}(x,y)\right). \end{aligned}$$

En mettant à gauche tous les termes en $Q(x,y)$, et en exprimant les fonctions $Q_{<i,<j}(x,y)$ comme des polynômes en les sections, on obtient une équation sous forme compacte :

$$K(x,y)Q(x,y) = x^{i_0}y^{j_0} + t\sum_{(i,j)\in\mathcal{S}} a_{i,j}(x,y)F_{i,j}(x,y) \tag{1.2.2}$$

où les $a_{i,j}(x,y)$ sont des polynômes de Laurent connus, les fonctions $F_{i,j}(x,y)$ sont des sections, et $K(x,y) = 1 - tS(x,y)$ est le noyau ‡. On est toujours en présence d'une équations aux variables catalytiques à *noyau*, soit une équation qui relie linéairement $Q(x,y)$ avec des fonctions inconnues ne faisant intervenir qu'une seule des deux variables.

À la différence des situations précédentes, l'équation précédente admet des cas dits « non-dégénérés » où la méthode du noyau unidimensionnel est inopérante. Un cas particulier notable intervient lorsque l'on se restreint aux petits pas, c'est-à-dire lorsque $\mathcal{S} \subset \{-1,0,1\}^2$. L'équation s'écrit alors de la manière suivante :

$$\begin{aligned} xyK(x,y)Q(x,y) = xy &- ty([x^{-1}]S(x,y))Q(0,y) - tx([y^{-1}]S(x,y))Q(x,0) \\ &+ t([x^{-1}y^{-1}]S(x,y))Q(0,0). \end{aligned} \tag{1.2.3}$$

Quand l'ensemble de pas $\mathcal{S}$ contient des pas qui reculent dans les deux directions (i.e. $[x^{-1}]S(x,y) \neq 0$ et $[y^{-1}]S(x,y) \neq 0$), on observe que l'équation (1.2.3) fait intervenir à la fois $Q(0,y)$ et $Q(x,0)$, aussi les deux variables catalytiques sont a priori nécessaires. Ces modèles de pas ont été l'objet d'une classification systématique. Cette première classification des chemins discrets dans le quadrant qui s'étend grosso modo de 2010 (article [BM10]) à 2018 (fin de la classification des modèles hypertranscendants, avec [DHRS18; DHRS21]) a interconnecté de très nombreux mathématiciens de disciplines très diverses, de combinatoire énumérative [BM10; Bou16], probabilités [FIM99; DW15; KR12], calcul formel [BK10; BCvKP17] et théorie de Galois des équations aux différences [DHRS21; DHRS18].

‡. La forme explicite du membre de droite dans le cas général n'apporte que peu d'intuition sur le chemin, aussi on s'est contenté ici de donner la forme grossière de l'équation. Elle est de toute manière suffisamment simple à retrouver à la main dans des cas particuliers.

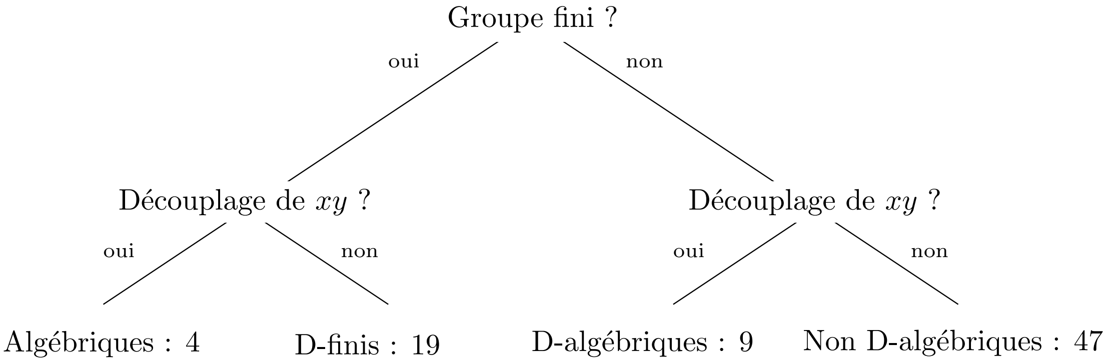


FIGURE 1.2.5 – Classification des 79 modèles à petits pas fondamentaux. La signification des conditions *groupe fini* et *découplage de xy* sera expliquée au chapitre suivant.

Depuis l'aboutissement de cette première classification, la progression naturelle consiste à étudier d'autres types de chemins via ces équations catalytiques, en adaptant les outils qui ont été développés dans le cadre de cette classification. C'est dans ce contexte très riche que se situe cette thèse.

1.3 Organisation du manuscrit

Le manuscrit de thèse présente deux travaux autour du dénombrement des marches dans le quadrant, issus en grande partie d'articles rédigés durant les trois ans de thèse. Aussi, le contenu de ces articles en anglais sera en grande partie reproduit, faisant ainsi l'économie d'un travail de traduction. Par suite, le vocabulaire des chemins dans le quadrant qui y est employé sera également introduit dans cette langue dans le chapitre suivant.

Les outils intervenant dans l'énumération des chemins dans le quadrant sont encore jeunes, et constamment redéfinis. Ce chaos se condensera peut-être dans quelques années avec une théorie générale permettant de tout compter. L'auteur n'a pas la prétention de fournir une telle théorie, aussi la présentation qui sera faite des différents outils fera apparaître des redondances. Chaque présentation d'une notion sera rattachée au texte dont elle est issue.

1.3.1 Chemins à petits pas dans le quart de plan

Le chapitre 2 rappelle les notions fondamentales utilisées lors de l'étude des chemins à petits pas dans le quart de plan. Notamment, il présente deux approches qui ne sont pas tout à fait équivalentes pour étudier l'équation fonctionnelle (1.2.3). La première approche présentée est « géométrique », et est due à [FIM99]. La seconde approche présentée, et qui s'en inspire, est « formelle », et est due à [BM10]. Ces deux approches, qui ont en commun de définir un certain groupe de transformations de l'équation, sont appliquées dans des cas différents. On fera usage plus loin dans la thèse de ces deux approches, et en profite pour introduire les notions nécessaires de théorie des corps.

1.3.2 Marches à grands pas

Mon premier travail a été de rendre possible l'application d'une stratégie d'algébricité pour les marches à grands pas (qui existait pour les petits pas, voir la section 2.2.2). En particulier, cette stratégie d'algébricité requerrait l'étude et la construction d'*invariants rationnels* et de *découplage* de fractions rationnelles.

L'objet d'un premier article avec Charlotte Hardouin a été d'étudier sous un prisme galoisien un graphe de changements de variables, appelé *l'orbite* et introduit dans [BBM21]. On montre d'abord au chapitre 3 qu'un certain groupe d'automorphismes finiment engendré agit transitivement et fidèlement sur l'orbite. Cela permet d'étudier plus finement ses symétries, et donne alors, lorsque cette orbite est finie, des moyens systématiques de construire des invariants rationnels. En outre, au chapitre 4, le formalisme de l'homologie des graphes et son interaction avec ledit groupe permettent de construire une obstruction au découplage de fractions rationnelles. Munis de ces deux outils, on construit explicitement des découplages et invariants pour diverses familles de chemins au chapitre 5. On prouve alors d'une part en section 5.2 des conjectures d'algébricité énoncées dans [BBM21], et on trouve d'autre part en section 5.3 une famille infinie de modèles à pas arbitrairement grands potentiellement algébriques.

1.3.3 Marches à bords interactifs

Mon second travail au chapitre 6 a consisté à étudier des chemins à petits pas dans le quart de plan, mais en considérant la statistique supplémentaire du nombre de contacts avec les deux axes, qualifiés de *chemins à bords interactifs*. En se concentrant sur une certaine famille de pas (dits *de genre zéro*, voir le chapitre idoine), on a pu appliquer les techniques de traitement des équations fonctionnelles pour ces modèles issues de [DHRS20]. On donne alors la classification complète de ces modèles, selon les poids a et b associés aux interactions avec les axes. En étudiant de manière effective les critères pour les solutions de ces équations, on découvre certaines relations inattendues entre les poids pour lesquelles la série génératrice est algébrique, tout en démontrant que dans tous les autres cas celle-ci n'est pas différentiellement algébrique.

Chapter 2

Small steps

In this chapter, we recall some known facts on the enumeration of quadrant walks with small steps, which will be reused or adapted in the remaining of the manuscript. We will in particular use this presentation to illustrate the mathematical notions required for understanding the manuscript.

The chapter is articulated around a fundamental object attached to a model with small steps, called the *group of the walk*. In the study of walks with small steps, this group was defined and used in two slightly different ways. Recall that as explained in Section 1.2.3, given a weighted model of walks with small steps, the functional equation for the generating function $Q(x,y)$ of walks in the quadrant starting at $(0,0)$ is

$$\begin{aligned}\widetilde{K}(x,y)Q(x,y) &= xy - ty([x^{-1}]S(x,y))Q(0,y) \\ &\quad - tx([y^{-1}]S(x,y))Q(x,0) \\ &\quad + t([x^{-1}y^{-1}]S(x,y))Q(0,0)\end{aligned} \tag{2.0.1}$$

with $\widetilde{K}(x,y) = xy(1 - tS(x,y))$. The goal is to obtain information on the series $Q(x,y)$ by exploiting the structure of the above functional equation. One possible way would be to extend the kernel method that was seen in Section 1.2.2. Two difficulties arise.

First, contrasting with the kernel method, the zero locus of $\widetilde{K}(x,y)$ is an algebraic curve, making necessary the analysis of the valuations in t of $x(y,t)$, for $x(y,t)$ a particular branch of the equation $\widetilde{K}(x(y,t),y) = 0$. Hence, it is not obvious how one would make the evaluation of the equation on all points of the curve $\widetilde{K}(x,y) = 0$ well defined.

Moreover, assuming that such an evaluation is well defined, the left-hand side of the equation would vanish, leaving the following equation

$$\begin{aligned}0 &= x(y,t)y - ty([x^{-1}]S(x,y))Q(0,y) \\ &\quad - tx(y,t)([y^{-1}]S(x(y,t),y))Q(x(y,t),0) \\ &\quad + t([x^{-1}y^{-1}]S(x,y))Q(0,0).\end{aligned}$$

Note that this equation has still three unknown functions. In all the cases, a group of transformations of the equation, the *group of the walk*, will be introduced to help isolate these functions.

The two approaches listed below both answer these two questions in different ways, to eventually solve the equation (2.0.1). The choice of the technique to use is highly dependent on the studied model.

In Section 2.1, we present a first approach, originated in [FIM99] and geometric in nature. For a fixed t, it sees the zero locus of $\widetilde{K}(x,y) = 0$ as a complex projective curve $\overline{E_t}$ admitting some nice parametrization $P \in \mathcal{C} \mapsto (x(P), y(P)) \in \overline{E_t}$, so that the convergence of all the functions in the equation is ensured when replacing (x,y) by $(x(P), y(P))$ for some of the points P. It then presents the group evoked earlier as a group of automorphisms of $\mathcal{C}$ which encodes the symmetries of the equation. This group induces a *difference operator* on the meromorphic functions on the kernel curve, which permits to see one of the unknown functions as the solution to some difference equation relative to this operator. This equation can be handled through Galois theory, or differential Galois theory of difference equations, depending on the finiteness of the group.

In Section 2.2 we show another approach and definition of the group, introduced in [BM10], which was adapted from the earlier geometric definition. Here, the kernel polynomial does not vanish. Instead, the group is more directly seen as a group of rational transformations of the indeterminates x and y, so that the polynomial $S(x,y)$ is constant on these substitutions. The group is then directly applied to the functional equation (1.2.3). This approach, more "formal" in nature, was the one that first admitted a direct extension to walks with large steps, for it does not require to describe the structure of the zero locus of $\widetilde{K}(x,y)$ (the geometric approach might extend for large steps [FR15]). Among the contributions of this approach are *orbit sums*, that allow for a simplification of the functional equation, and give algebraicity and some D-finiteness proofs. As an example, we present an adaptation of the so-called Tutte invariant method [Tut95].

2.1 Geometric approach

The first systematic approach to walks in the quadrant was developped in the book by Fayolle, Iasnogorodski and Malyshev [FIM99], from a probabilistic point of view. In their context, the functional equations do not concern the generating function of walks, but rather the generating function of the stationary distribution of a random walk. Nevertheless, their techniques are general enough so that they were applied in the enumerative combinatorics context of the counting of walks in the quadrant, as presented below.

2.1.1 Kernel curve

So as to remove the only unknown of the equation depending on both variables x and y (the function $Q(x,y)$), and following the kernel method described earlier, it is natural to make vanish the *kernel* polynomial

$$\widetilde{K}(x,y) \stackrel{\text{def}}{=} xy\,(1 - tS(x,y)) \in \mathbb{C}[x,y,t].$$

For every complex number t, the kernel polynomial $\widetilde{K}(x,y)$ defines a complex affine curve in $\mathbb{C}^2$

$$E_t \stackrel{\text{def}}{=} \{(x,y) \in \mathbb{C} \times \mathbb{C} \,:\, \widetilde{K}(x,y) = 0\}.$$

One considers the projective closure of this curve in $\mathbb{P}^1 \times \mathbb{P}^1$ *, which we define now. Here, $\mathbb{P}^1 = \mathbb{P}^1(\mathbb{C})$ denotes the complex projective line, which may be defined as the quotient of $\mathbb{C} \times \mathbb{C}$ by the equivalence relation

$$(x_0, x_1) \sim (x_0', x_1') \iff \exists \lambda \in \mathbb{C}^\star, (\lambda x_0, \lambda x_1) = (x_0', x_1').$$

One denotes by $[x_0 : x_1]$ the class of (x_0, x_1). The affine line $\mathbb{C}$ embeds into $\mathbb{P}^1$ through the map $x \mapsto [x : 1]$, and we thus denote $\infty \stackrel{\text{def}}{=} [1 : 0]$ (the projective line $\mathbb{P}^1$ can be thought of as the complex plane with an additional point at ∞). The projective closure of E_t in $\mathbb{P}^1 \times \mathbb{P}^1$ is then

$$\overline{E_t} \stackrel{\text{def}}{=} \left\{([x_0 : x_1], [y_0 : y_1]) \in \mathbb{P}^1 \times \mathbb{P}^1 \,:\, x_1^2 y_1^2 \widetilde{K}(\tfrac{x_0}{x_1}, \tfrac{y_0}{y_1}) = 0\right\}.$$

In [DHRS21], the authors examine study the geometric properties of the curve $\overline{E_t}$ for weighted models with small steps. We summarize it in the following proposition. A weighted model is called nondegenerate when t the model $\mathcal{S}$ has at least one step in each direction, so that $[x^{-1}]S(x,y)$, $[y^{-1}]S(x,y)$, $[x^1]S(x,y)$ and $[y^1]S(x,y)$ are all nonzero.

Proposition 2.1.1. *When the weighted model described by $S(x,y)$ is nondegenerate, then the kernel polynomial $\widetilde{K}(x,y)$ is irreducible for all $t > 0$ as an element of $\mathbb{C}[x,y]$. Hence, the curve $\overline{E_t}$ is* irreducible*, and admits a birational parametrization $\phi : \mathcal{C} \to \overline{E_t}$. Two cases happen.*

1. *Either $\mathcal{C} = \mathbb{P}^1$, this case called the* genus 0 case*. Indeed, the genus of $\mathbb{P}^1$ is 0 (informally, it has no hole).*
2. *Either there exist two non-colinear complex numbers ω_1 and ω_2 such that $\mathcal{C} = \mathbb{C}/\Lambda$ with $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$, this case this time called the* genus 1 case*. Indeed, the surface $\mathbb{C}/\Lambda$ has genus 1 (informally, it has one hole).*

For uniformity in the treatment, we for now forget about the specifics of $\mathcal{C}$. We simply say that it is an irreducible smooth projective curve, along with a birational map $\phi : \mathcal{C} \to \overline{E_t}$. Moreover, since the curve $\overline{E_t}$ is a subset of $\mathbb{P}^1 \times \mathbb{P}^1$, we may compose ϕ with the two projections of $\mathbb{P}^1 \times \mathbb{P}^1$ onto its first and second factor, and we end up with two rational maps $x : \mathcal{C} \to \mathbb{P}^1$ and $y : \mathcal{C} \to \mathbb{P}^1$. Thus, in the remaining of the text, we will denote a point of the curve $\overline{E_t}$ by a tuple $(x(P), y(P))$, with P a point of $\mathcal{C}$.

*. We consider the projective closure, for this curve has better geometric and topological properties (in particular, $\overline{E_t}$ is a compact Riemann surface).

2.1.2 Function field and Galois theory

In algebraic geometry, any complex irreducible projective curve X is endowed with what is called its *function field* denoted $\mathfrak{K}(X)$. It corresponds to the field of all *rational maps* $X \to \mathbb{P}^1$. The precise definition of rational maps, and a nice introduction to classical algebraic geometry may be found in Chapter 1 of [Har77], or in Appendix B of [Sti09] for the projective case. In the case of the curve $\mathcal{C}$ (with $\widetilde{K}(x,y)$ irreducible), with $x : \mathcal{C} \to \mathbb{P}^1$ and $y : \mathcal{C} \to \mathbb{P}^1$ being the projection maps defined earlier, the function field of $\mathcal{C}$ is

$$\mathfrak{K}(\mathcal{C}) = \mathbb{C}(x,y),$$

the rational maps x and y satisfying the relation $\widetilde{K}(x,y) = 0$.

Any automorphism of the complex curve $\mathcal{C}$ will induce a $\mathbb{C}$-algebra automorphism of its function field $\mathfrak{K}(\mathcal{C})$, *et vice-versa* [Sti09, Appendix B]. For instance, for the first direction, if $\tau : \mathcal{C} \to \mathcal{C}$ is an automorphism, then τ induces a $\mathbb{C}$-algebra automorphism $\widetilde{\tau}$ of $\mathfrak{K}(\mathcal{C})$ given by

$$\widetilde{\tau}(f)(P) \overset{\text{def}}{=} f(\tau P) \text{ for every rational map } f : \mathcal{C} \to \mathbb{P}^1.$$

Conversely, if σ is a $\mathbb{C}$-algebra automorphism of $\mathfrak{K}(\mathcal{C})$, it is possible to find a corresponding automorphism τ of $\mathcal{C}$ satisfying $\widetilde{\tau} = \sigma$. When τ is an automorphism of $\mathcal{C}$ and $f : \mathcal{C} \to \mathbb{P}^1$ a rational map (equivalently an element of $\mathfrak{K}(\mathcal{C})$), then we will write

$$f^{\tau} \overset{\text{def}}{=} \widetilde{\tau}(f).$$

The reason for the exponential notation is because the action of automorphisms of $\mathcal{C}$ on the field $\mathfrak{K}(\mathcal{C})$ is a right action, meaning that

$$f^{(\tau_1\tau_2)} = (f^{\tau_1})^{\tau_2}.$$

Arithmetic of the function field of the curve

We now give additional terminology on function fields. We first recall some terminology and basic results on fields.

Definition 2.1.2 (Chapter I of [Sza09])**.** Let k, L, M be fields.

1. Consider a ring homomorphism $\iota : k \to L$. Then ι is injective. Thus, k embeds in L. We then identify k with the image of its embedding in L through this morphism, and we write it $k \subset L$ (thus forgetting the *structure embedding*). We then say that L is an *extension* of k, and write it L/k.
2. For M/k and L/k two extensions, a ring homomorphism $\sigma : M \to L$ is a *k-algebra homomorphism* if $\sigma(x) = x$ for all $x \in k$ (i.e. σ fixes k). For two extensions L and M of k, we denote by $\mathrm{Hom}_k(M,L)$ the set of k-algebra homomorphisms from M to L. We also denote by $\mathrm{Aut}_k(L)$ the group of k-algebra automorphisms of L.

3. The *degree* of an extension L/k is the dimension of L as a k-vector space, and is denoted $[L : k]$. When the degree is finite, the extension L/k is called *finite*. If M/L and L/k are extensions, then M/k is an extension, and one has $[M : k] = [M : L] \cdot [L : k]$.
4. Let L/k be an extension. An element $x \in L$ is algebraic (over k) if the extension $k(x)/k$ is finite. If all the elements of L are algebraic, then the extension L/k is called *algebraic*.

We often consider some base field k. For this base field, we may find some large algebraic extension containing all the roots of polynomials in $k[Z]$: the *algebraic closure* of k.

Proposition 2.1.3 (Chapter I of [Sza09])**.** *Let k be a field.*

1. *A field is* algebraically closed *if it has no other algebraic extension than itself. An* algebraic closure *of k is an algebraically closed field $\overline{k}$ such that the extension $\overline{k}/k$ is algebraic.*
2. *There exists an algebraic closure $\overline{k}$ of k. It is unique up to a k-algebra isomorphism.*
3. *For an algebraic extension L/k, there exists a k-algebra homomorphism $\sigma : L \to \overline{k}$.*
4. *If an extension L/k is algebraic, any k-algebra homomorphism $\sigma : L \to \overline{k}$ extends to a k-algebra isomorphism $\widetilde{\sigma} : \overline{L} \to \overline{k}$. As a consequence, $\overline{k} \simeq \overline{L}$, and $\overline{k}$ is an algebraic closure of L.*
5. *For L/k an algebraic extension, the restriction homomorphism*

$$\begin{aligned} \mathrm{Hom}_k(\overline{k}, \overline{k}) &\longrightarrow \mathrm{Hom}_k(L, \overline{k}) \\ \sigma &\longmapsto \sigma_{|L} \end{aligned}$$

is a surjection, with kernel $\mathrm{Hom}_L(\overline{k}, \overline{k})$.

We now recall basic facts on *divisors* of a function field of an algebraic curve. A comprehensive introduction to these notions is contained in [Sti09].

Definition/Proposition 2.1.4 (Chapter I of [Sti09])**.** Consider the curve $\mathcal{C}$, with function field $\mathfrak{K}(\mathcal{C})$. A zero of a fonction $h \in \mathfrak{K}(\mathcal{C})$ is a point P of $\mathcal{C}$ such that $h(P) = 0$. A *divisor on* $\mathcal{C}$ is a formal finite sum of points of $\mathcal{C}$, that is $D = \sum_{P \in \mathcal{C}} n_P \cdot P$ where the n_P are integers. The *degree* of a divisor $D = \sum_P n_P P$ of $\mathcal{C}$ is defined as $\deg D = \sum_P n_P$. The map $D \mapsto \deg D$ is a group homomorphism from the group of divisors to $\mathbb{Z}$. The following properties hold:

1. Let h be a nonzero function in $\mathfrak{K}(\mathcal{C})$. The function h has finitely many zeros in $\mathcal{C}$. The *zero divisor* of h is thus defined as

$$(h)_0 = \sum_{P \text{ zero of } h} \mathrm{ord}_P(h) \cdot P$$

where $\mathrm{ord}_P(h)$ is the multiplicity of P as a zero of h. Similarly, the *polar divisor* $(h)_\infty$ of h is defined as the zero divisor of h^{-1}.

2. The *principal divisor* associated to a nonzero function h is defined as

$$(h) = (h)_0 - (h)_\infty.$$

It has the property that $(h) = 0$ if and only if h is belongs to $\bar{k} \cap \mathfrak{K}(\mathbb{C})$ (h is a constant).

3. For u, v two nonzero functions in $\mathfrak{K}(\mathcal{C})$, then $(1/u) = -(u)$ and $(uv) = (u) + (v)$.

4. For $h \in \mathfrak{K}(\mathcal{C})$ not a constant, the following holds:

$$\deg(h)_0 = \deg(h)_\infty = [\mathfrak{K}(\mathcal{C}) : \mathbb{C}(h)].$$

Moreover, $1 \leq [\mathfrak{K}(\mathcal{C}) : \mathbb{C}(h)] < \infty$ (recall that for L/k an extension, $[L : k]$ denotes the dimension of L as a k-vector space).

Example 2.1.5. Consider the complex curve $\mathbb{P}^1$ and the rational map $s : \mathbb{P}^1 \to \mathbb{P}^1$ defined by $s(P) = P$ for all P in $\mathbb{P}^1$. Then the function field of $\mathbb{P}^1$ is $\mathfrak{K}(\mathbb{P}^1) = \mathbb{C}(s)$, the field of complex fractions in one variable.

Let $f(s) \in \mathbb{C}(s)$ be a rational map. Then $f(s) = \frac{u(s)}{v(s)}$ for some relatively prime polynomials $u(Z), v(Z)$ in $\mathbb{C}[Z]$. If $P \in \mathbb{C}$ is a zero of $f(s)$, then $u(P) = 0$. Indeed, since u and v are relatively prime, Bézout identity yields that $a(Z)u(Z) + b(Z)v(Z) = 1$ for some $a(Z)$, $b(Z)$ in $\mathbb{C}[Z]$, so replacing Z with $s(P)$ in the above equation shows that $u(s(P))$ and $v(s(P))$ cannot both be zero. Since $\mathbb{C}$ is algebraically closed, the sum of the multiplicities of the zeros of $u(s)$ in $\mathbb{C}$ is equal to $\deg_Z u(Z)$. The point $P = \infty$ is a zero of $f(s)$ if $\deg_Z u(Z) < \deg_Z v(Z)$, and then its multiplicity is equal to $\deg_Z v(Z) - \deg_Z u(Z)$. Thus, the degree of the zero divisor of $f(s)$ is

$$\deg\,(f)_0 = \max\{\deg_Z u(Z), \deg_Z v(Z)\}$$

We now compute the degree of the extension $\mathbb{C}(s)/\mathbb{C}(f(s))$. The function s is a root of the polynomial

$$\mu(Z) := u(Z) - h(s)v(Z) \in \mathbb{C}[h(s), Z].$$

Since $h(s) \neq 0$, this polynomial has degree $\max\{\deg_Z u(Z), \deg_Z v(Z)\}$. Moreover, it is irreducible. Since $h(s)$ is transcendental over $\mathbb{C}$, we may see $\mu(Z)$ as an element of the polynomial ring $\mathbb{C}[h(s), Z]$. Since $\mu(Z)$ has degree 1 in $h(s)$, it is irreducible in $\mathbb{C}(Z)[h(s)]$. Moreover, if $\mu(Z) = a(Z)\widetilde{\mu}(Z, h(s))$, for some nonconstant $a \in \mathbb{C}[Z]$, then one of the roots z_0 of $\mu(Z)$ belongs to $\mathbb{C}$. Then $u(z_0) = h(s)v(z_0)$. If $v(z_0) = 0$, then $u(z_0) = 0$, which is impossible since $u(Z)$ and $v(Z)$ have no common roots. If $v(z_0) \neq 0$, this implies that $h(s)$ belongs to $\mathbb{C}$, which is impossible. Therefore, $\mu(Z)$ has content 1, so Gauss Lemma [Lan02, Chapter IV, theorem 2.3] implies that $\mu(Z)$ is irreducible in $\mathbb{C}[h(s), Z]$. In particular, $\mu(Z)$ is irreducible in $\mathbb{C}(h(s))[Z]$. This shows that $[\mathbb{C}(s) : \mathbb{C}(f(s))] = \deg(f)_0$, and by the same argument that $[\mathbb{C}(s) : \mathbb{C}(f(s))] = \deg(f)_\infty$. ∎

Galois theory

We recall here basic notions of Galois theory. We consider a base field k of characteristic 0 for the sake of simplicity (meaning that $\mathbb{Q}$ embeds into k). In particular, the following definitions are not given in the most general setting to ease understanding. All the definitions and propositions found in this section can be found in Chapter I of [Sza09].

Definition/Proposition 2.1.6 (Galois extension). For L/k an algebraic extension, we denote $\mathrm{Aut}_k(L)$ the group of k-algebra automorphisms of L. For H a subgroup of $\mathrm{Aut}_k(\overline{k})$, denote $L^H \subset L$ the subfield of elements fixed by H, i.e.

$$L^H \overset{\mathrm{def}}{=} \{x \in L : \forall \sigma \in H, \sigma(x) = x\}.$$

Note that we have always $k \subset L^{\mathrm{Aut}_k(L)}$. When $k = L^{\mathrm{Aut}_k(L)}$, then the extension L/k is called *Galois* (the elements of k are exactly those fixed by $\mathrm{Aut}_k(L)$). The group $\mathrm{Aut}_k(L)$ is then called the *Galois group* of the extension, and is often denoted $\mathrm{Gal}(L/k)$.

There is an easy criterion to determine when an extension L/k is Galois. Recall from Proposition 2.1.3 that if L/k is algebraic, then $\overline{k}$ is an algebraic closure of L.

Proposition 2.1.7 (Chapter I [Sza09]). *Let L/k be an algebraic extension. It is called* normal *if for any k-algebra homomorphism $\sigma \in \mathrm{Aut}_k(\overline{k})$, then $\sigma(L) \subset L$. If k has characteristic zero, then L/k is normal if and only if it is Galois.*

Example 2.1.8. When k has characteristic 0, then the extension $\overline{k}/k$ is Galois. Its Galois group G_k is called the *absolute Galois group of k*. ■

Example 2.1.9. Assume that L/k has degree 2. Then L/k is Galois. Indeed, it is straightforward to see that there exists x in L with $L = k(x)$, having minimal polynomial $P(X) = X^2 - a$ for a in k and a not a square (consider any element α in $L \setminus k$, we may take x to be the discriminant of the minimal polynomial of α over k). The polynomial $P(X)$ factors in L as $(X - x)(X + x)$. Therefore, if σ is a k-algebra automorphism of $\overline{k}$, then $P(\sigma x) = 0$, therefore $\sigma x = \pm x$. It follows that L/k is normal, hence Galois. Its Galois group has order 2, generated by the k-algebra homomorphism ι_1 induced by $x \mapsto -x$. ■

Theorem 2.1.10 (Fundamental theorem of finite Galois theory). *Let M/k be a Galois extension, and let $G = \mathrm{Gal}(M/k)$ its Galois group. Then there is an anti-equivalence of categories (called the* Galois correspondence*) between the intermediate extensions of L/k and the subgroups G ordered by inclusion, given by the following two maps:*

1. *Let L/k be an intermediate extension of M/k. Then the extension M/L is Galois. We may thus consider its Galois group $H \overset{\mathrm{def}}{=} \mathrm{Gal}(M/L)$, which is a subgroup of G.*
2. *Let H be a subgroup of G, then we consider the following subfield of M: $L \overset{\mathrm{def}}{=} M^H = \{f \in M : \text{for all } \tau \in H, \tau(f) = f\}$, which contains k, so L/k is an intermediate extension of M/k.*

Moreover, $[M:k] = \#G$.

Lemma 2.1.11. *Let L/k be a Galois extension and $\mu \in k[Z]$ an irreducible polynomial with some root α in L. Then $\mu(Z)$ splits in L with distinct roots ($\mu(Z) = (X-x_1)\dots(X-x_k)$ with x_i in L), and the group $\mathrm{Gal}(L/k)$ acts transitively on its roots.*

Example 2.1.12. Consider $x_1, x_2, \dots, x_n$ to be n indeterminates over the rational numbers $\mathbb{Q}$. Consider the polynomial

$$\begin{aligned} P(X) &\stackrel{\text{def}}{=} (X-x_1)(X-x_2)\dots(X-x_n) \\ &= X^n - e_{n-1}X^{n-1} + \dots + (-1)^n e_0. \end{aligned}$$

One has $[X^i]P(X) = (-1)^{n-i}e_i$ for $i < n$, with e_i the i-th elementary symmetric functions on the variables x_i. Then the extension $L/k \stackrel{\text{def}}{=} \mathbb{Q}(x_1,\dots,x_n)/\mathbb{Q}(e_0,\dots,e_{n-1})$ is algebraic. Moreover, it is immediately seen to be normal, hence Galois. We denote its Galois group $G \stackrel{\text{def}}{=} \mathrm{Gal}(L/k)$. Given a permutation $\sigma \in S_n$, with S_n the symmetric group on n elements, one can define a $\mathbb{Q}$-algebra automorphism as follows

$$f(x_1,\dots,x_n) \in \mathbb{Q}(x_1,\dots,x_n) \mapsto f^\sigma(x_1,\dots,x_n) \stackrel{\text{def}}{=} f(x_{\sigma(1)},\dots,x_{\sigma(n)}).$$

Since the e_i are symmetric functions, this map is a k-algebra automorphism. Thus, the group S_n embeds in G. Conversely, any element of G induces a permutation on the roots x_i of $P(X)$, thus this embedding is a group isomorphism.

Now, let $f(x_1,\dots,x_n) \in L$ be a function satisfying $f^\sigma = f$ for all $\sigma \in S_n$. Then by Theorem 2.1.10, f must belong to $k = \mathbb{Q}(e_{n-1},\dots,e_0)$. Thus, Galois theory is a generalization of the theory of symmetric functions. ∎

This presentation of Galois theory is far from from the original ideas of Galois, and took almost a century to be given this exposition which is due to Artin. The interested reader may look at [Kie71] for a historical perspective. Such presentation is equally distant to the modern functorial presentation of Grothendieck [Sza09].

2.1.3 Group of the walk

It has been established in [FIM99] that the curve $\mathcal{C}$ given by Proposition 2.1.1 is equipped with two involutive automorphisms ι_1 and ι_2. They are defined so that for all $P \in \mathcal{C}$ one has

$$x(\iota_1 P) = x(P) \quad \text{and} \quad y(\iota_2 P) = y(P). \tag{2.1.1}$$

We look for such automorphisms for the following (informal) reason. Assume for a moment that for fixed $t > 0$, the series $Q(x(P), y(P))$, $Q(x(P), 0)$ and $Q(0, y(P))$ converge for P in $U \subset \mathcal{C}$ an (analytic) open subset of $\mathcal{C}$. Then by substituting $(x(P), y(P))$ for (x, y) in the equation (1.2.3), we are lead to consider an equation

$$0 = x(P)y(P) + A(x(P)) + B(y(P)), \tag{2.1.2}$$

with

$$A(x) = -tx([y^{-1}]S(x,y))Q(x,0) + t([x^{-1}y^{-1}]S(x,y))Q(0,0)$$
$$B(y) = -ty([x^{-1}]S(x,y))Q(0,y)$$

Now, if we assume that U is stable under ι_1, then we can through the same argument obtain an equation

$$0 = x(P)y(\iota_1 P) + A(x(P)) + B(y(\iota_1 P)), \tag{2.1.3}$$

since $x(\iota_1 P) = x(P)$. We can then eliminate $A(x(P))$ between (2.1.2) and (2.1.3), to obtain

$$0 = x(P)(y(P) - y(\iota_1 P)) + B(y(P)) - B(y(\iota_1 P)). \tag{2.1.4}$$

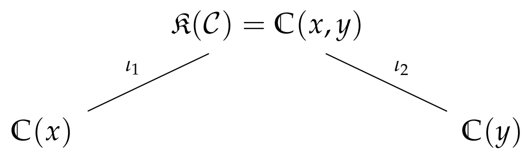
Note that the unknown function $A(x)$ disappeared, so we are left with a nontrivial equation on the series $B(y)$. This process justifies that we change one variable at a time, hence the requirements of (2.1.1).

This being said, let us see how are defined ι_1 and ι_2. Recall that $\mathfrak{K}(\mathcal{C}) = \mathbb{C}(x,y)$ with $x : \mathcal{C} \to \mathbb{P}^1$ and $y : \mathcal{C} \to \mathbb{P}^1$ the projection maps.

Proposition 2.1.13. *1. The extension $\mathfrak{K}(\mathcal{C})/\mathbb{C}(x)$ has degree 2. Thus, it is Galois, with Galois group generated by an involution ι_1.*

2. Similarly, the extension $\mathfrak{K}(\mathcal{C})/\mathbb{C}(y)$ has degree 2. Thus, it is Galois, with Galois group generated by an involution ι_2.

The following lattice serves as a summary.



Proof. In virtue of the fact that the set of steps $\mathcal{S}$ has at least one step in each direction, the polynomial $\widetilde{K}(X,Y)$ satisfies $\deg_X \widetilde{K}(X,Y) = \deg_Y \widetilde{K}(X,Y) = 2$. Since it is irreducible in both variables, and that $\widetilde{K}(x,y) = 0$, one has $[\mathfrak{K}(\mathcal{C}) : \mathbb{C}(x)] = [\mathfrak{K}(\mathcal{C}) : \mathbb{C}(y)] = 2$. It thus follows that these extensions are Galois (see Example 2.1.9), which serves to define ι_1 and ι_2. □

In virtue of the correspondence between automorphisms of the function field, the maps ι_1 and ι_2 induce automorphisms of the curve $\mathcal{C}$, which satisfy (2.1.1). In particular, the two following equations hold:

$$\begin{aligned} y(\iota_1(P)) &= \frac{([y^{-1}]S(x,y))(x(P))}{y(P)([y]S(x,y))(x(P))} \\ x(\iota_2(P)) &= \frac{([x^{-1}]S(x,y))(y(P))}{x(P)([x]S(x,y))(y(P))}. \end{aligned} \tag{2.1.5}$$

The *group of the walk* G is then defined as the group of automorphisms of $\mathcal{C}$ generated by ι_1 and ι_2. These two involutions induce an automorphism σ of $\mathcal{C}$ defined as

$$\sigma \stackrel{\text{def}}{=} \iota_2 \circ \iota_1. \tag{2.1.6}$$

Depending on the order of σ, two situations happen.

1. If σ has finite order n, then the group of the walk admits the following presentation:
$$\left\langle a, b \mid a^n = 1, b^2 = 1, ba = a^{-1}b \right\rangle,$$
(r corresponding to σ and t to ι_1). Thus, the group is isomorphic to the dihedral group of order $2n$.
2. In the case where σ has infinite order, then the group of the walk is infinite, admitting the following presentation:
$$\left\langle a, b \mid b^2 = 1, ba = a^{-1}b \right\rangle.$$

We finish with one example that uses the above terminology.

Example 2.1.14. Consider $k_{\text{inv}} \stackrel{\text{def}}{=} \mathbb{C}(x) \cap \mathbb{C}(y)$, $\mathbb{K} = \overline{\mathfrak{K}(\mathcal{C})}$, and $\sigma \stackrel{\text{def}}{=} \iota_2\iota_1$. If the group $G = \langle \iota_1, \iota_2 \rangle$ is finite (equivalently $\sigma^n = 1$ for some $n > 0$), then consider the polynomial

$$P(Z) = \prod_{k=0}^{n-1}(Z - \sigma^k x) \cdot \prod_{k=0}^{n-1}(Z - \iota_1\sigma^k x).$$

This polynomial is nonconstant, and by construction its coefficients are fixed by G. In particular, they are fixed by ι_1 and ι_2, generators of the Galois groups of $k(\mathcal{C})/\mathbb{C}(x)$ and $k(\mathcal{C})/\mathbb{C}(y)$. Therefore, since these extensions are Galois, the coefficients of $P(Z)$ lie in k_{inv}, hence the extension $\mathbb{C}(x)/k_{\text{inv}}$ is algebraic since x is a root of P. Since the transcendence degree of $\mathbb{C}(x)/\mathbb{C}$ is equal to 1, the field k_{inv} is not equal to $\mathbb{C}$, hence there exists a nonconstant function $f(x) = g(y) \in \mathbb{C}(x) \cap \mathbb{C}(y)$.

Conversely, if such an $f(x) = g(y)$ exists, then $\mathbb{C}(x)/\mathbb{C}(f(x))$ and $\mathbb{C}(y)/\mathbb{C}(g(y))$ are finite. Since $f(x) = g(y)$ is fixed by G, then so are conjugates of x and y. Therefore, the set $\{(\tau x, \tau y) : \tau \in G\}$ is finite, so is G since the functions x and y generate $k(\mathcal{C})$.

Thus, if G is infinite, the solutions to $h^\sigma = h$ are $h \in \mathbb{C}$. ■

2.1.4 Working on the equation

Let us finish this section by giving some perspective on how to apply this group to the study of the functional equation (1.2.3) for walks with small steps. We assume here that the group G is infinite.

Recall that the algebraic curve $\mathcal{C}$ can be seen as a compact Riemann surface. Depending on the genus of $\mathcal{C}$, it is either the Riemann sphere $\mathbb{P}^1$ (genus 0), or a torus $\mathbb{C}/\Lambda$ (genus 1), with $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ for some non colinear complex numbers ω_i. It can be shown that the group has the following expression on the points of $\mathcal{C}$ (see [DHRS21]):

— In the genus 0 case where $\mathcal{C} = \mathbb{P}^1$, there exists a complex number q with $|q| \neq 1$ such that for all $s \in \mathbb{P}^1$ one has

$$\iota_1(s) = \frac{1}{s} \qquad\qquad \iota_2(s) = \frac{q}{s}.$$

— In the genus 1 case where $\mathcal{C} = \mathbb{C}/\Lambda$, there exists a complex number $\omega_3 \in \mathbb{C}^*$ such that

$$\iota_1(s) = -s \qquad\qquad \iota_2(s) = \omega_3 - s.$$

Using these explicit expressions, it is easy to prove that in both cases there exists some affine open set $U \subset \mathcal{C}$ such that $0 \in U \cap \sigma U$. Moreover, the open set satisfies that for $W \in \{U, \iota_2 U, \sigma U\}$, and $s \in W$, the power series $Q(x(s), y(s))$, $Q(x(s), 0)$ and $Q(0, y(s))$ are convergent for all $|t| < 1$. The hereby defined functions of s are then analytic on W.

Thus, in the functional equation

$$\begin{aligned}\widetilde{K}(x,y)Q(x,y) = xy &- tya(y)Q(0,y)\\ &- txb(x)Q(x,0)\\ &+ t\varepsilon Q(0,0),\end{aligned}$$

where

$$a(y) \stackrel{\text{def}}{=} [x^{-1}]S(x,y) \qquad b(x) \stackrel{\text{def}}{=} [y^{-1}]S(x,y)\text{and} \qquad \varepsilon \stackrel{\text{def}}{=} [x^{-1}y^{-1}]S(x,y),$$

one may replace (x, y) by $(x(s), y(s))$ for $s \in U$, thus allowing us to obtain

$$\begin{aligned}0 = x(s)y(s) &- ty(s)a(y(s))Q(0,y(s))\\ &- tx(s)b(x(s))Q(x(s),0)\\ &+ t\varepsilon Q(0,0).\end{aligned} \tag{2.1.7}$$

One may also replace (x, y) by $(x(\iota_2 s), y(\iota_2 s)) = (x^{\iota_2}(s), y^{\iota_2}(s))$ for $s \in U$, thus allowing us to obtain

$$\begin{aligned}0 = x^{\iota_2}(s)y(s) &- ty(s)a(y(s))Q(0,y(s))\\ &- tx^{\iota_2}(s)b(x^{\iota_2}(s))Q(x^{\iota_2}(s),0)\\ &+ t\varepsilon Q(0,0),\end{aligned} \tag{2.1.8}$$

(since $y^{\iota_2}(s) = y(s)$ for all s in $\mathcal{C}$). We can thus eliminate $Q(0, y(s))$ between (2.1.7) and (2.1.8), to find an equation valid for all s in U

$$tx^{\iota_2}(s)b(x^{\iota_2}(s))Q(x^{\iota_2}(s),0) = tx(s)b(x(s))Q(x(s),0) + (x^{\iota_2}(s) - x(s))y(s). \tag{2.1.9}$$

We then define $V \stackrel{\text{def}}{=} U \cap \sigma U$, and an analytic function

$$\breve{F}(s) \stackrel{\text{def}}{=} x(s)b(x(s))Q(x(s),0)$$

for all s in V. Since

$$x^{\iota_2}(s) = (x^{\iota_1})^{\iota_2}(s) = x^{\sigma^{-1}}(s) = x(\sigma^{-1}(s))$$

and that $\sigma^{-1}(s) \in U$ for $s \in V$, one gets from (2.1.9) the following equation, valid for all $s \in V$:

$$\breve{F}(\sigma^{-1}(s)) = \breve{F}(s) + (x^{\iota_2}(s) - x(s))y(s). \tag{2.1.10}$$

This relation is then used to find a meromorphic continuation of $\breve{F}$ on $\mathbb{C}$, denoted $\widetilde{F}$. This step is straightforward in genus 0 (see [DHRS20] or Section 6.1.3) and trickier in genus 1 [KR12]. The equation (2.1.10), called a *difference equation* can then be handled through a Galois theory of difference equations [Ish98; HS08; DR19]. In the classification of walks with small steps, this theory is used for distinguishing between D-algebraic and non-D-algebraic models.

2.2 Formal approach

In 2010, Bousquet-Mélou and Mishna reopened the systematic investigation of walks in the quadrant [BM10]. Inspired by the geometric definition of the group of the walk, they designed a formal definition, only acting on pairs (x, y) with x and y the algebraically independent variables of the functional equation. Thus, their group directly acts on the coefficients in t of formal power series in $\mathbb{C}(x, y)[[t]]$, allowing to perform simplification on the functional equation without requiring convergence. Their approach gives paths for proofs of algebraicity and D-finiteness, through the notion of *orbit sums*.

2.2.1 A formal group of the walk

Rather than canceling the kernel $\widetilde{K}(x, y) = xy(1 - tS(x, y))$, one rather wants to find substitutions on the pair (x, y) so that $S(x, y)$ remains constant. Considering (2.1.5), one defines two maps Ψ and Φ playing the same role as ι_1 and ι_2, but acting directly on pairs of algebraically independent variables. Consider two elements u, v belonging to the field $\mathbb{K} \overset{\text{def}}{=} \mathbb{C}(x, y)$. Then one defines maps Ψ and Φ as follows. We require that $S(u, v) = S(\Psi(u, v))$ and $S(u, v) = S(\Phi(u, v))$, and that when applying Φ or Ψ, at most one coordinate changes, so that for all $(u, v) \in \mathbb{K} \times \mathbb{K}$ one has

$$\Psi(u, v) = (u, v') \text{ for some } v' \qquad \Phi(u, v) = (u', v) \text{ for some } u'.$$

Since $\deg_x \widetilde{K}(x, y) = \deg_y \widetilde{K}(x, y) = 2$, this forces to define these maps as

$$\begin{aligned} \Psi : \mathbb{K} \times \mathbb{K} &\longrightarrow \mathbb{K} \times \mathbb{K} \\ (u, v) &\longmapsto \left(u, \frac{([y^{-1}]S(x, y))(u)}{v\,([y^1]S(x, y))(u)}\right) \end{aligned} \qquad \begin{aligned} \Phi : \mathbb{K} \times \mathbb{K} &\longrightarrow \mathbb{K} \times \mathbb{K} \\ (u, v) &\longmapsto \left(\frac{([x^{-1}]S(x, y))(v)}{u\,([x^1]S(x, y))(v)}, v\right). \end{aligned} \tag{2.2.1}$$

The *formal group of the walk* $\boldsymbol{G}$ is then defined as the subgroup of permutations of $\mathbb{K} \times \mathbb{K}$ generated by Ψ and Φ.

In the study of this group, one often considers the orbit of the pair of algebraically independent variables (x, y) under this group. The finiteness of this orbit is equivalent to the finiteness of the group. Indeed, if the orbit is finite, then in particular the orbit of (x, y) under the subgroup $\langle \Psi\Phi \rangle$ must also be finite, so $(\Psi\Phi)^n(x, y) = (x, y)$ for some $n \geq 1$. But then, by substitution (recall that x and y are variables), this implies that $(\Psi\Phi)^n(f, g) = (f, g)$ for any pair $(f, g) \in \mathbb{K} \times \mathbb{K}$. Therefore, $(\Psi\Phi)^n = 1$. Since $\Psi^2 = \Phi^2 = 1$, this implies that the formal group of the walk $\boldsymbol{G}$ is finite, for every element must be of the form $(\Psi\Phi)^k$ or $\Phi(\Psi\Phi)^k$ for some integer k. For instance, the orbit for the Kreweras model (see [Kre65]) is finite, as Figure 2.2.1 below demonstrates.

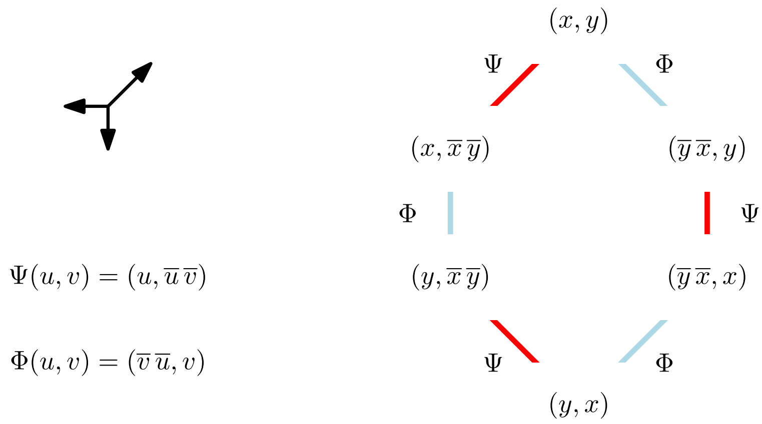


Figure 2.2.1 – The Kreweras model and its orbit

2.2.2 The Tutte invariants method

In this section, we present one application of the formal group of the walk to the classification of models of walks with small steps, which is a technique to prove the algebraicity of $Q(x, y)$, based on the notion of *Tutte invariants*. We fix some weighted model $\mathcal{S}$ with small steps, and we assume that the formal group of the walk $\boldsymbol{G}$ is finite.

Equations in one catalytic variable

In [BJ06], Bousquet-Mélou and Jehanne proved the algebraicity of power series solution of *well founded* polynomial equations in one catalytic variable. Their method has been further extended recently to the case of systems of divided difference equations by Notarantonio and Yurkevich in [NY23]. These algebraicity results are in fact particular cases of an older result in commutative algebra of Popescu [Pop86] but the strength of the strategy developed in [BJ06; NY23] lies in the effectiveness of the approach of the latter.

Let $\mathbb{L}$ be a field of characteristic zero. For an unknown bivariate function $F(u,t)$ denoted for short $F(u)$, we consider the functional equation

$$F(u) = F_0(u) + t\,Q\left(F(u), \Delta F(u), \Delta^{(2)}F(u), \dots, \Delta^{(k)}F(u), t, u\right), \tag{2.2.2}$$

where $F_0(u) \in \mathbb{L}[u]$ is given explicitly and Δ is the *discrete derivative*: $\Delta F(u) = \frac{F(u)-F(0)}{u}$. One can easily show that the equation (2.2.2) has a unique solution $F(u,t)$ in $\mathbb{L}[u][[t]]$, the ring of formal power series in t with coefficients in the ring $\mathbb{L}[u]$. Such an equation is called *well-founded*. Here is one of the main results of [BJ06].

Theorem 2.2.1 (Theorem 3 in [BJ06]). *The formal power series $F(u,t)$ defined by (2.2.2) is algebraic over $\mathbb{L}(u,t)$.*

We shall use Theorem 2.2.1 as a black box in order to establish the algebraicity of power series solutions of a polynomial equation in one catalytic variable.

Tutte invariants: from two variables to one

We now present a method inspired by Tutte [Tut95] which was adapted by Bernardi, Bousquet-Mélou and Raschel to small steps walks [BBR21] and by Bousquet-Mélou to three quadrant walks [Bou23]. We reproduce here the method of [Bou23] which relies on a suitable notion of t-invariants and an *Invariant Lemma* for multivariate power series. The strategy developed in [Bou23] is an adaptation for formal power series of the approach already introduced in Section 4.3 in [BBR21].

Definition 2.2.2. We denote by $\mathbb{C}(x,y)((t))$ the field of Laurent series in t with coefficients in the field $\mathbb{C}(x,y)$. We denote by $\mathbb{C}_{\text{mul}}(x,y)((t))$ the subring of $\mathbb{C}(x,y)((t))$ formed by series of the form

$$H(x,y) = \sum_n \frac{p_n(x,y)}{a_n(x)b_n(y)} t^n,$$

where $p_n(x,y) \in \mathbb{C}[x,y]$, $a_n(x) \in \mathbb{C}[x]$ and $b_n(y) \in \mathbb{C}[y]$.

Definition 2.2.3 (Definition 2.4 in [Bou23]). Let $H(x,y) \in \mathbb{C}_{\text{mul}}(x,y)((t))$ be a Laurent series. The series $H(x,y)$ is said to have *poles of bounded order at* 0 if for some natural numbers m and n, the series $x^m y^n H(x,y)$ belongs to $\mathbb{C}[x,y]((t)) \subset \mathbb{C}[\![x,y,t]\!]$.

Given a model $\mathcal{S}$, one can use the notion of poles of bounded order at 0 to construct an equivalence relation in the ring $\mathbb{C}_{\text{mul}}(x,y)((t))$. To this purpose, we slightly adapt Definition 2.5 in [Bou23] to encompass the large steps case.

Definition 2.2.4 (t-equivalence). Let $F(x,y)$ and $G(x,y)$ be two Laurent series in $\mathbb{C}_{\text{mul}}(x,y)((t))$. We say that these series are *t-equivalent* (with respect to $\widetilde{K}(x,y)$), and we write $F(x,y) \equiv G(x,y)$ if the series $\frac{F(x,y)-G(x,y)}{K(x,y)}$ has poles of bounded order at 0.

The t-equivalence is compatible with the ring operations on Laurent series applied pairwise as stated below.

Proposition 2.2.5 (Lemma 2.5 in [Bou23])**.** *If $A(x,y) \equiv B(x,y)$ and $A'(x,y) \equiv B'(x,y)$, then $A(x,y) + B(x,y) \equiv A'(x,y) + B'(x,y)$ and $A(x,y)B(x,y) \equiv A'(x,y)B'(x,y)$.*

The notion of t-equivalence allows us to define the notion of t-invariants as follows.

Definition 2.2.6 (t-Invariants (Definition 2.3 in [Bou23]))**.** Let $I(x)$ and $J(y)$ be two Laurent series in t with coefficients lying respectively in $\mathbb{C}(x)$ and $\mathbb{C}(y)$. If $I(x) \equiv J(y)$, then the pair $(I(x), J(y))$ is said to be a *pair of t-invariants* (with respect to the model $\mathcal{S}$).

By Proposition 2.2.5, pairs of t-invariants are also preserved under sum and product applied pairwise. We now state the main result on t-invariants [Bou23, Lemma 2.6] whose proof originally for small steps models passes directly to the large steps context.[†].

Lemma 2.2.7 (Invariant Lemma)**.** *Let $(I(x), J(y))$ be a pair of t-invariants. If the coefficients in the t-expansion of $\frac{I(x)-J(y)}{K(x,y)}$ are all multiples of xy, then there exists a Laurent series $A(t)$ with coefficients in $\mathbb{C}$ such that $I(x) = J(y) = A(t)$.*

Note that each of the equations $I(x) = A(t)$ and $J(y) = A(t)$ involves only one catalytic variable. In other words, the Invariant Lemma 2.2.7 allows us to produce nontrivial equations with one catalytic variable from one pair of t-invariants satisfying a certain analytic regularity.

Of course, one actually needs to find suitable pairs of invariants involving the sections $Q(x,0)$ and $Q(0,y)$, so that $I(x) = J(y) = A(t)$ gives nontrivial equations on these sections. This will be done through the search for *decouplings* and *rational invariants*, whose construction in [BBR21] exploits the finiteness of the group in a crucial manner.

Decoupling

A first pair of invariants can be found by looking at the functional equation:

$$\begin{aligned}\widetilde{K}(x,y)Q(x,y) &= xy - ty([x^{-1}]S(x,y))Q(0,y)\\ &\quad - tx([y^{-1}]S(x,y))Q(x,0) + t([x^{-1}y^{-1}]S(x,y))Q(0,0).\end{aligned}$$

Notice that the functional equation has the form

$$\widetilde{K}(x,y)Q(x,y) = xy - A(x) - B(y) \tag{2.2.3}$$

†. In [Bou23], Lemma 2.6 requires that the coefficients in the t-expansion of $\frac{I(x)-J(y)}{K(x,y)}$ vanish at $x = 0$ and $y = 0$. This is equivalent to the condition stated in Lemma 2.2.7.

with

$$\begin{aligned} A(x) &= tx([y^{-1}]S(x,y))Q(x,0) - t([x^{-1}y^{-1}]S(x,y))Q(0,0) \\ \text{and } B(y) &= ty([x^{-1}]S(x,y))Q(0,y). \end{aligned}$$

If one manages to write the pair xy as

$$xy \equiv f(x) + g(y) \tag{2.2.4}$$

for $f(x)$ in $\mathbb{C}(x,t)$ and $g(y)$ in $\mathbb{C}(y,t)$ two fractions, then from (2.2.3) and Proposition 2.2.5, one has the following t-equivalence (because it is obvious that $Q(x,y)$ has poles of bounded order at 0, for its coefficients belong to $\mathbb{C}[x,y]$):

$$f(x) - A(x) \equiv B(y) - g(y).$$

Thus,

$$(I_1(x), J_1(y)) = (f(x) - A(x), B(y) - g(y))$$

forms a pair of t-invariants, where the components $I_1(x)$ and $J_1(y)$ depend on the sections $Q(x,0)$ and $Q(0,y)$. The condition to form this first pair of t-invariants was the writing of xy as in (2.2.4). This motivates this first definition:

Definition 2.2.8. Let $H(x,y)$ be a fraction in $\mathbb{C}(x,y,t)$. If there exist fractions $F(x) \in \mathbb{C}(x,t)$ and $G(y) \in \mathbb{C}(y,t)$ such that

$$H(x,y) \equiv F(x) + G(y),$$

then we say that $H(x,y)$ admits a t-decoupling.

In [BBR21, Theorem 4.11], the authors describe how to determine if a given arbitrary fraction admits a t-decoupling, and if it exists how to give such a decoupling. This method requires the group of the walk $\boldsymbol{G}$ to be finite, so that $(\Psi\Phi)^n = 1$. If $f(x) \in \mathbb{C}(x,t)$, let $f(u,v) \overset{\text{def}}{=} f(u)$. Then since $\Psi(u,v)$ and (u,v) share the same coordinate for any pair $(u,v) \in \mathbb{K} \times \mathbb{K}$, one has

$$\sum_{k=0}^{n-1} f((\Phi\Psi)^k(x,y)) - f((\Psi(\Phi\Psi)^k)(x,y)) = 0.$$

Similarly, for $g(y) \in \mathbb{C}(y,t)$, and setting $g(u,v) \overset{\text{def}}{=} g(v)$, then one also computes

$$\sum_{k=0}^{n-1} g((\Phi\Psi)^k(x,y)) - g((\Psi(\Phi\Psi)^k)(x,y)) = 0.$$

Therefore, if $h(x,y) \in \mathbb{C}(x,y,t)$ admits a t-decoupling with

$$h(x,y) = f(x) + g(y) + \widetilde{K}(x,y)r(x,y),$$

$$\text{then } \sum_{k=0}^{n-1} h((\Phi\Psi)^k(x,y)) - h((\Psi(\Phi\Psi)^k)(x,y)) \\ = \frac{\widetilde{K}(x,y)}{xy} \sum_{k=0}^{n-1} (xyr)((\Phi\Psi)^k(x,y)) - (xyr)((\Psi(\Phi\Psi)^k)(x,y)),$$

for $\frac{\widetilde{K}(x,y)}{xy}$ is constant on the pairs (u,v) in the orbit of (x,y) under the action of $\boldsymbol{G}$. Thus, if the factor of $\widetilde{K}(x,y)$ in the right-hand side has poles of bounded order at $x=0$ and $y=0$, then one must have

$$\sum_{k=0}^{n-1} h((\Phi\Psi)^k(x,y)) - h((\Psi(\Phi\Psi)^k)(x,y)) \equiv 0.$$

From [BBR21, Theorem 4.11], this last line actually characterizes the fact that h admits a t-decoupling. The same theorem gives a way to obtain the x and y components of the t-decoupling through some other well chosen sum. These linear combinations of evaluations of functions on the pairs of the orbit of (x,y) under the action of $\boldsymbol{G}$ are called *orbit sums*.

Rational invariants

Another source of invariants that we will use is the following:

Definition 2.2.9. A pair of t-invariants $(F(x),G(y))$ with $F(x)\in\mathbb{C}(x,t)$ and $G(y)\in\mathbb{C}(y,t)$ is called a pair of *rational t-invariants*. If $F(x)$ or $G(y)$ does not belong to $\mathbb{C}(t)$, then the pair is called *non-trivial*.

It is natural to look for such t-invariants once a first pair $(I_1(x),J_1(y))$ involving the sections $Q(x,0)$ and $Q(0,y)$ has been found, since our goal is to ultimately find a polynomial equation on $I_1(x)$ and $J_1(y)$ through the use of Lemma 2.2.7. We thus hope to eliminate the poles of $(I_1(x),J_1(y))$ using an additional pair of rational t-invariants.

As in the previous paragraph, the rational t-invariants may be obtained through the construction of well chosen orbit sums. It has been proved in [BBR21, Theorem 4.6] that when the formal group of the walk $\boldsymbol{G}$ is finite, then such a pair exists automatically. The corresponding orbit sum consists in starting from a random fraction $H(x,y)$ and to average it over the pairs of the orbit, i.e. to compute

$$H_\sigma(x,y) \stackrel{\text{def}}{=} \sum_{(u,v)\in\boldsymbol{G}\cdot(x,y)} H(u,v). \tag{2.2.5}$$

Then $H_\sigma(x,y)\equiv I_2(x)$ and $H_\sigma(x,y)\equiv J_2(y)$ for some $I_2(x)\in\mathbb{C}(x,t)$ and $J_2(y)\in\mathbb{C}(y,t)$.

From these two pairs of invariants, the goal is to hopefully combine them using Proposition 2.2.5 to form a pair of t-invariants satisfying the condition of Lemma 2.2.7, and eventually apply the machinery of Theorem 2.2.1 to prove the algebraicity of both $Q(x,0)$ and $Q(0,y)$, and finally the algebraicity of $Q(x,y)$.

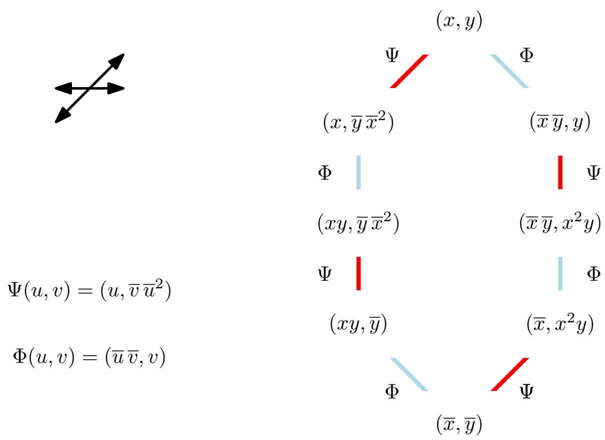


Figure 2.2.2 – The orbit of the Gessel model

Example: algebraicity of the Gessel walks

Here, we present an explicit application of the above strategy on the Gessel walks (cf. Figure 2.2.2), whose generating function has been proved to be algebraic using this method but with a different notion of invariants ‡. This proof originates in [BBR21].

The generating function $Q(x,y)$ of the Gessel model satisfies the following functional equation, put in normal form:

$$\begin{aligned} \widetilde{K}(x,y)Q(x,y) &= xy - t(y+1)Q(0,y) - tQ(x,0) + tQ(0,0) \\ \text{with } \widetilde{K}(x,y) &= xy - t(x^2y^2 + x^2y + y + 1). \end{aligned} \tag{2.2.6}$$

The group of the walk for the Gessel model is finite (Figure 2.2.2). Thanks to [BBR21, Proposition 4.11], we may rewrite the fraction xy as

$$xy = -\frac{1}{x} + \frac{y}{t(y+1)} - \frac{\widetilde{K}(x,y)}{tx(y+1)}.$$

Therefore, reinjecting this expression inside (2.2.6), and putting on the left-hand side everything having $\widetilde{K}(x,y)$ as a factor, we obtain the following first pair of t-invariants:

$$P_1 \stackrel{\text{def}}{=} (I_1(x), J_1(y)) = \left(\frac{1}{x} + A(x), \frac{y}{t(y+1)} - B(y)\right). \tag{2.2.7}$$

with $A(x) = tQ(x,0) - tQ(0,0)$ and $B(y) = t(y+1)Q(0,y)$.

We can also exploit the finiteness of the group of the walk to produce a pair of rational t-invariants. Computing the orbit sum $(x)_\sigma$ (see (2.2.5)) gives a satisfying result, yielding the following pair of t-invariants

$$P_2 \stackrel{\text{def}}{=} (I_2(x), J_2(y)) \equiv \left(-\frac{1}{x^2} + \frac{1}{tx} + 2 + \frac{x}{t} - x^2, \frac{(1+y)^2}{y} + \frac{y}{t^2(1+y)^2}\right). \tag{2.2.8}$$

‡. What characterizes a good notion of invariants is the existence of a lemma similar to Lemma 2.2.7.

One notices that $J_1(y) + B(0)$ has a simple zero at $y = 0$, that $J_2(y)$ has a simple pole at $y = 0$, so that $(J_1(y) + B(0)) \cdot J_2(y)$ has no pole at $y = 0$. This forms the second component of the pair of invariants

$$P_3 \stackrel{\text{def}}{=} (P_1 + B(0)) \cdot P_2, \tag{2.2.9}$$

(the products and sum being taken componentwise).

It remains to eliminate the pole at $x = 0$ of the first component. The first component of P_3 expands near 0 as

$$I_3(x) = -\frac{1}{x^3} + O(1/x^2),$$

while the first component of P_1 expands as

$$I_1(x) = \frac{1}{x} + O(1).$$

Therefore, one might hope to eliminate the pole at $x = 0$ of order 3 of $I_3(x)$ using $I_1(x)$. Indeed, one shows that

$$\begin{aligned} I_4(x) &\stackrel{\text{def}}{=} I_3(x) + I_1(x)^3 - (2A(0) + \frac{1}{t} - B(0))I_1(x)^2 \\ &- \left(2 + \frac{A(0)+B(0)}{t} + 2A'(0) + 3A(0)^2 - 2(2A(0) + \frac{1}{t} - B(0))A(0)\right) I_1(x) \end{aligned} \tag{2.2.10}$$

has no pole at $x = 0$. Note that $J_3(y) = \frac{1}{t} - B'(0) + O(y)$, and that $J_1(y) = B(0) + O(y)$, so that any polynomial combination of $J_3(y)$ and $J_1(y)$ has no pole at $y = 0$, so this holds in particular for the series $J_4(x)$ defined as the second component of the pair P_4, defined as

$$\begin{aligned} P_4 &\stackrel{\text{def}}{=} P_3 + {P_1}^3 - (2A(0) + \frac{1}{t} - B(0)){P_1}^2 \\ &- (2 + \frac{A(0)+B(0)}{t} + 2A'(0) + 3A(0)^2 - 2\left(2A(0) + \frac{1}{t} - B(0)\right) A(0))P_1. \end{aligned} \tag{2.2.11}$$

Finally, one checks that

$$\frac{I_4(x) - J_4(y)}{\widetilde{K}(x,y)}$$

has no pole at $x = 0$ nor at $y = 0$. Indeed, we have the following lemma:

Lemma 2.2.10. *Consider $S(x,y)$ a Laurent polynomial with denominator $x^a y^b$, with $a, b > 0$. Consider $K(x,y) = 1 - t\,S(x,y)$, and take $I(x) \equiv J(y)$ be a pair of invariants. Assume also that $I(x)$ and $J(y)$ belong respectively to $\mathbb{C}[\![x]\!]((t))$ and $\mathbb{C}[\![y]\!]((t))$. Then the coefficients of $G(x,y) = \frac{I(x)-J(y)}{K(x,y)}$ are a multiple of $x^a y^b$.*

Proof. By assumption, since $I(x) \equiv J(y)$, we have that the series $x^k y^{k'} G(x,y)$ belongs to $\mathbb{C}[\![x,y]\!]((t))$ for some $k, k' \geq 0$.

In particular, $G(x,y)$ belongs to $\mathbb{C}((t,y))((x))$. We will thus expand with respect to the variable x both sides of the following identity:

$$I(x) - J(y) = K(x,y)G(x,y).$$

On the one hand, we have

$$G(x,y) = \lambda x^\alpha + O(x^{\alpha+1})$$

for $\lambda \neq 0$ a constant, and $\alpha \in \mathbb{Z}$ the valuation of $G(x,y)$ as a Laurent series in the variable x. Moreover, by assumption on $S(x,y)$, the expansion of $K(x,y)$ with respect to x is

$$K(x,y) = \mu x^{-a} + O(x^{-a+1})$$

for $\mu \neq 0$. Hence,

$$K(x,y)G(x,y) = \tau x^{\alpha-a} + O(x^{\alpha-a})$$

for $\tau \neq 0$ a constant.

On the other hand, since $I(x)$ has no pole at $x = 0$, we have

$$I(x) - J(y) = I(0) - J(y) + O(x).$$

Therefore, we deduce that $\alpha \geq a$, and thus that the coefficients in t of $G(x,y)$ are multiples of x^a. Similarly, one shows that the coefficients in t of $G(x,y)$ are multiples of y^b. □

Thus the pair of invariants P_4 satisfies the conditions of Lemma 2.2.7. Thus, we obtain two polynomial equations in one catalytic variable, respectively on $Q(x,0)$ (upon which depends $A(x)$) and $Q(0,y)$ (upon which depends $B(y)$):

$$I_4(x) = I_4(0) \tag{2.2.12}$$
$$J_4(x) = I_4(0). \tag{2.2.13}$$

One then checks that the induced equations

$$Pol_1(Q(x,0), x, t, A(0), A'(0), B(0), I_4(0)) = 0 \tag{2.2.14}$$
$$Pol_2(Q(0,y), y, t, A(0), A'(0), B(0), I_4(0)) = 0 \tag{2.2.15}$$

are well founded (see Section 2.2.2).

Therefore, Theorem 2.2.1 ensures that $Q(x,0)$ and $Q(0,y)$ are algebraic. Moreover, the method of [BJ06] gives explicit polynomial equations on $Q(x,0)$ and $Q(0,y)$, so that the number of Gessel walks may be computed explicitly.

Chapter 3

Galois structure on the orbit and invariants

We saw in the previous chapter the tools used in the classification of weighted walks with small steps. Namely, two slightly different groups have been introduced for studying the functional equation in two catalytic variables. The first is the *classical group of the walk* of Section 2.1, seen as a group of automorphisms of the kernel curve. The second is the *formal group of the walk* of Section 2.2, seen as group of rational transformations of $\mathbb{C}(x,y) \times \mathbb{C}(x,y)$ that fix $S(x,y)$. Both definitions of those groups required that the weighted model had small steps.

— The construction of the *classical group of the walk* required that for $x : \mathcal{C} \to \mathbb{P}^1$ and $y : \mathcal{C} \to \mathbb{P}^1$, both extensions $\mathbb{C}(x,y)/\mathbb{C}(x)$ and $\mathbb{C}(x,y)/\mathbb{C}(y)$ were Galois, which was automatic for they had degree 2.

— Similarly, the construction of the *formal group of the walk* required that the nontrivial solution u' of the equation $S(u',v) = S(u,v)$ (or v' of the equation $S(u,v') = S(u,v)$) was rational in terms of u and v, true when the steps are small.

In [BBM21], the question of extending such tools to the study of the generating function of walks based on models with arbitrarily large steps was raised. Since none of the aforementioned conditions exist, the group could not be directly defined as above. If we try to define the kernel curve $\overline{E_t}$ as the projective closure of the affine curve $\widetilde{K}(x,y) = 0$ for a fixed t, its structure is not as nice. For instance, the proofs that the kernel polynomial $\widetilde{K}(x,y)$ was irreducible for a fixed $t > 0$ were ad-hoc, relying on its low degree. Therefore, the approach that was taken by Bostan, Bousquet-Mélou and Melczer in [BBM21] was to extend the formal group of the walk, by finding substitutions (u,v) of the variables (x,y), leaving $S(u,v)$ unchanged, and constructed starting from (x,y) by changing at most one coordinate at a time. The induced graph, called the *orbit* (which we present below in more detail), is named this way because for small steps it corresponds to the orbit of the pair (x,y) under the group $\langle \Phi, \Psi \rangle$. However, in the general case, the transition from one coordinate to another is not induced by a rational transformation.

Some techniques that used the formal group of the walk could be extended to the orbit when it is finite, mainly orbit sums. Indeed, consider the following example, taken from [BBM21].

Example 3.0.1 (Proposition 5.2, [BBM21])**.** Consider the model whose Laurent polynomial is $S(x,y) = x + x^{-1} + x^{-2}y + y^{-1}$. Writing $K(x,y) = 1 - tS(x,y)$, the generating function of walks starting at $(0,0)$ based on the underlying set of steps $\mathcal{S}$ satisfies the following functional equation:

$$\begin{aligned} K(x,y)Q(x,y) &= 1 - tx^{-1}Q(0,y) - tyx^{-2}Q(0,y) - tyx^{-2}([x^1]Q(x,y)) - ty^{-1}Q(x,0) \\ &= x^{-2}y^{-1}(x^2y - ty^2Q(0,y) - ty^2([x^1]Q(x,y)) - tx^2Q(x,0). \end{aligned} \tag{3.0.1}$$

The authors of [BBM21] then compute the orbit as in Figure 3.0.1 below.

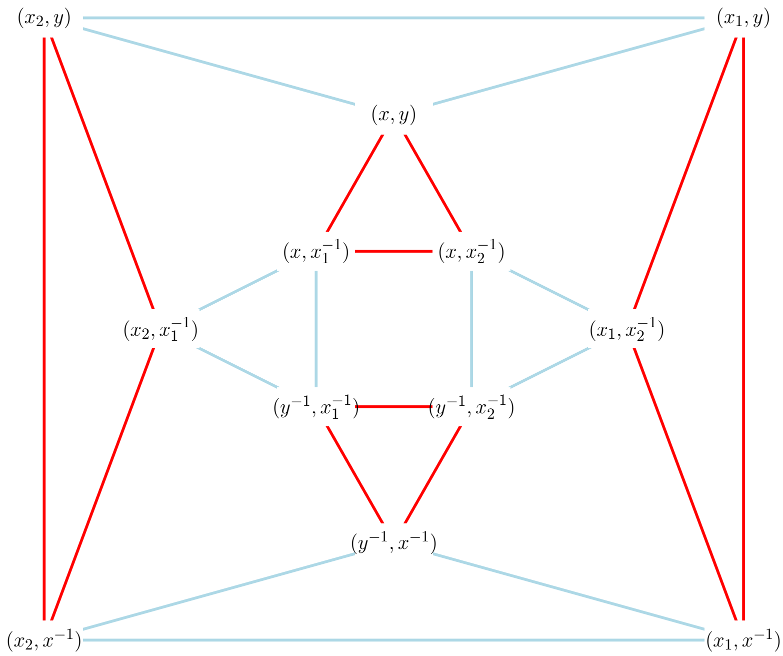


Figure 3.0.1 – The orbit of the model $\mathcal{S}$ is finite.

They then evaluate (3.0.1) on pairs of the orbit, and then perform linear combinations of these equations. They obtain an equation

$$(x^2yQ(x,y) + \sum_{i,j} \lambda_{i,j}x_i^2y_jQ(x_i,y_j)) = \frac{R(x,y)}{K(x,y)} + \sum_{i,j} R_{i,j} \tag{3.0.2}$$

with (x_i, y_j) pairs of the orbit, $\lambda_{i,j}, R_{i,j} \in \mathbb{C}(x, x_1, x_2, y, y_1, y_2)$, and $R(x,y) \in \mathbb{C}(x,y)$ explicit. They extract the *nonnegative part* of both sides of the above equations. The nonnegative part $[x^{\geq}y^{\geq}]F(x,y)$ of $F(x,y) = x^{-n}y^{-m}G(x,y)$ for $G(x,y) \in \mathbb{C}[[x,y]]$ is defined as

$$[x^{\geq}y^{\geq}]F(x,y) = \sum_{i\geq n, j\geq m} x^{-n}y^{-m}[x^i y^j]G(x,y)$$

and is a power series in x and y by construction. This extends coefficientwise for a series $F(x,y) \in x^{-n}y^{-m}\mathbb{C}[[x,y,t]]$.

The nonnegative part extraction yields for this model the equation

$$x^2yQ(x,y) = [x^{\geq}y^{\geq}]\frac{R(x,y)}{K(x,y)}.$$

This shows for instance that the series $Q(1,1)$ is $D-finite$, as the diagonal of a rational power series. ∎

However, these orbit sums are found "by hand", for the orbit lacks structure. In this chapter, we show how one can associate to a weighted model $\mathcal{S}$ a group, which we call *the group of the walk*, generated by Galois automorphisms of two field extensions. We prove that the group of the walk acts faithfully and transitively on the orbit, analogously to the classical group. When the orbit is finite, this group is itself presented as a Galois group. We use this group and its interaction with the orbit to study some forms of invariants and decouplings, respectively in Section 3.2 of the present chapter, and in Chapter 4. We apply these two constructions to extend the algebraicity proof seen in Section 2.2.2 to walks with arbitrarily large steps, in Chapter 5.

3.1 Group acting on the orbit

From now on, we fix $S(x,y) \in \mathbb{C}(x,y)$, and assume the following

Assumption 3.1.1. *The fraction $S(x,y) \in \mathbb{C}(x,y)$ is non-univariate.*

We denote by k the field $\mathbb{C}(S(x,y))$, and

$$\widetilde{K}(x,y) = A(x,y) - tB(x,y).$$

where $S(x,y) = \frac{A(x,y)}{B(x,y)}$ for $A(x,y)$, $B(x,y)$ two relatively prime polynomials in $\mathbb{C}[x,y]$ (in other words, $\widetilde{K}(x,y)$ is the numerator of $1 - tS(x,y)$).

In particular, one may note that for any weighted model $\mathcal{S}$ with both positive and negative steps in each direction, assumption 3.1.1 holds (the other cases being one-dimensional are not interesting, as seen in Section 1.2.2).

In Section 3.1.1, we recall the definition of the orbit of a weighted model $\mathcal{S}$ with large steps. We give it a Galois structure in Section 3.1.2. In Section 3.1.3, we define the group of the walk and prove that it acts faithfully and transitively by graph automorphisms on the orbit. Finally, we investigate the evaluation of rational fractions in $\mathbb{C}(x,y,t)$ on the orbit.

3.1.1 The orbit

We recall below the definition of the orbit introduced in [BBM21, Section 3], and we also fix once and for all an algebraic closure $\mathbb{K}$ of $\mathbb{C}(x, y)$.

Definition 3.1.2 (Definition 3.1 in [BBM21])**.** Let (u, v) and (u', v') be in $\mathbb{K} \times \mathbb{K}$.

If $u = u'$ and $S(u, v) = S(u', v')$, then the pairs (u, v) and (u', v') are called *x-adjacent*, and we write $(u, v) \sim^x (u', v')$. Similarly, if $v = v'$ and $S(u, v) = S(u', v')$, then the pairs (u, v) and (u', v') are called *y-adjacent*, and we write $(u, v) \sim^y (u', v')$. Both relations are equivalence relations on $\mathbb{K} \times \mathbb{K}$.

If the pairs (u, v) and (u', v') are either x-adjacent or y-adjacent, they are called *adjacent*, and we write $(u, v) \sim (u', v')$. Finally, denoting by $\sim^*$ the reflexive transitive closure of $\sim$, the *orbit of the walk*, denoted by $\mathcal{O}$, is the equivalence class of the pair (x, y) under the relation $\sim^*$.

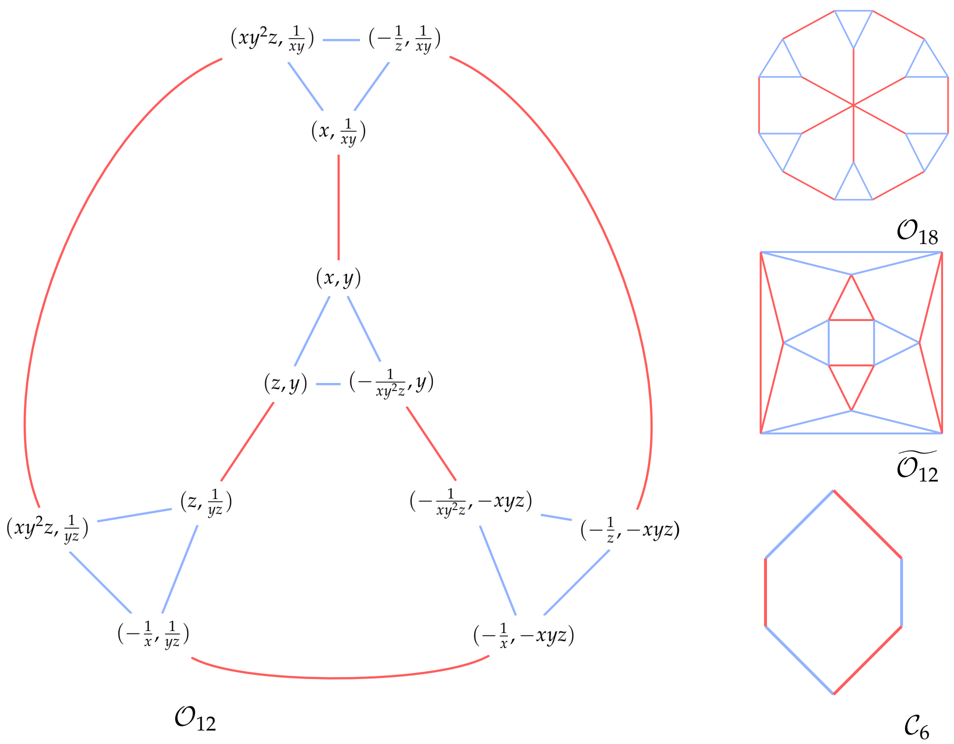


Figure 3.1.1 – A sample of finite orbits

The orbit $\mathcal{O}$ has a graph structure: the vertices are the pairs of the orbit and the edges are x,y-adjacencies, colored here by their adjacency type (note that we do not consider reflexive edges). The x-adjacencies are represented in red and the y-adjacencies in blue. As the x and y adjacencies come from equivalence relations, the monochromatic connected components of $\mathcal{O}$ are *cliques* (any two vertices of such a component are connected by an edge).

Moreover, by definition of the transitive closure, the graph $\mathcal{O}$ is *connected*, that is, every two vertices of the graph are connected by a path. In the sequel, we denote by $\mathcal{O}$ either the set of pairs in the orbit or the induced graph. The structure considered should be clear from the context. For a model $\mathcal{S}$, its *orbit type* corresponds to the class of its orbit under graph isomorphisms.

Section 10 in [BBM21] lists the distinct orbit types for models with steps in $\{-1,0,1,2\}^2$ with at least one large step. The orbit type being preserved when one reverses the model, they also correspond to the orbit types of models with steps in $\{-2,-1,0,1\}$. For these models, the finite orbit types are exactly $\mathcal{O}_{12}, \widetilde{\mathcal{O}_{12}}$ and $\mathcal{O}_{18}$ in Figure 3.1.1 and the cartesian product orbit-types of the *Hadamard models* that correspond to a step polynomial of the form $R(x)+P(x)Q(y)$ (see [BBKM16, Section 5] or [BBM21, Section 6] or Section 4.6.2).

Example 3.1.3 (Small steps)**.** For small steps models, the orbit when finite is always isomorphic to a cycle whose vertices all belong to $\mathbb{C}(x,y)^2$, for it corresponds exactly to the orbit of the pair (x,y) under the action of the formal group of the walk introduced in Section 2.2. Example $\mathcal{C}_6$ in Figure 3.1.1 is for instance the unlabelled orbit of the small steps model given by $S(x,y)=1/x+y+x/y$ [BM10, Example 2]. ■

Example 3.1.4 (The model $\mathcal{G}_\lambda$)**.** For the model $\mathcal{G}_\lambda=\{(-1,-1),(0,1),(1,-1),((1,0),\lambda),(2,1)\}$ the polynomial $\widetilde{K}(Z,y,1/S(x,y))$ is reducible over $k(x,y)[Z]$ and factors as

$$\widetilde{K}(Z,y,1/S(x,y)) = \frac{(Z-x)yP(Z)}{x^3y^2+(\lambda y+1)x^2+y^2x+1}$$
$$\text{where } P(Z)=x\,y^2Z^2+(x^2y^2+\lambda xy+x)Z-1.$$

Thus, an element $(z,y)\in\mathbb{K}^2$ distinct from (x,y) is y-adjacent to (x,y) if and only if z is a root of $P(Z)$. Its roots are of the form $z, -\frac{1}{xy^2z}$ by the relation between the roots and the coefficients of a degree two polynomial. One can then show that the orbit $\mathcal{O}_{12}$ in Figure 3.1.1 is the orbit of the model $\mathcal{G}_\lambda$, whatever the value of λ. ■

Before going on, let us discuss the finiteness of the orbit. For small steps walks, the finiteness of the orbit depends only on the order of $\Phi\circ\Psi$. Some number theoretic considerations on the torsion subgroup of the Mordell-Weil group of a rational elliptic surface prove that this order, when finite, is bounded by 6, which provides a very easy algorithm to test the finiteness of the group of the walk. This bound is valid for any choice of weights contained in an algebraically closed field of characteristic zero (see [HS08, Remark 5.1] and [SS19, Corollary 8.21]). For models with arbitrarily large steps, there is currently no general criterion to determine whether the orbit is finite or not, the only current way being to compute its pairs by saturation until it ends. There is a criterion [BBM21, Theorem 7] to prove that a given model has an infinite orbit that applies sometimes. This criterion, based on a fixed point argument, generalizes a criterion for small steps walks developed in [BM10]. We hope that, analogously to the small steps case, a geometric interpretation of the notion of orbit will provide some nice bounds on the potential diameter of the orbit and thus some efficient algorithms to test the finiteness of the orbit.

3.1.2 The Galois extension of the orbit

In the remaining of the manuscript, we denote by $k(\mathcal{O})$ the subfield of $\mathbb{K}$ generated over $k = \mathbb{C}(S(x,y))$ by all coordinates of the orbit $\mathcal{O}$. Note that $k(\mathcal{O})$ coincides with $\mathbb{C}(\mathcal{O})$ since x, y belong to the orbit.

The field $\mathbb{K}$ introduced in Section 3.1.1 is an algebraic closure of $\mathbb{C}(x,y)$. By definition of the orbit, $k(\mathcal{O}) = \mathbb{C}(\mathcal{O})$ is an algebraic field extension of $\mathbb{C}(x,y)$. Moreover, since y is algebraic over $k(x)$ and x is algebraic over $k(y)$, then $\mathbb{C}(x,y)$ is an algebraic field extension of $k(x)$ and $k(y)$. Therefore, $k(\mathcal{O})$ is algebraic over $k(x)$ and $k(y)$. Proposition 2.1.3 thus implies that $\mathbb{K}$ is an algebraic closure of $k(x), k(y)$ and $k(\mathcal{O})$.

We let any $\mathbb{C}$-algebra endomorphism σ of $\mathbb{K}$ act on $\mathbb{K} \times \mathbb{K}$ coordinate-wise by

$$\sigma \cdot (u,v) \overset{\text{def}}{=} (\sigma(u), \sigma(v)).$$

The following lemma establishes the compatibility of the equivalence relation $\sim^*$ with the action of $\mathbb{C}$-algebra endomorphisms of $\mathbb{K}$.

Lemma 3.1.5. *Let (u,v) and (u',v') be two pairs in $\mathbb{K} \times \mathbb{K}$ and $\sigma\colon \mathbb{K} \to \mathbb{K}$ be a $\mathbb{C}$-algebra endomorphism. Then $(u,v) \sim^x (u',v')$ (resp. $(u,v) \sim^y (u',v')$) implies that $\sigma \cdot (u,v) \sim^x \sigma \cdot (u',v')$ (resp. $\sigma \cdot (u,v) \sim^y \sigma \cdot (u',v')$). The same holds for $\sim^*$.*

Proof. Since σ is a $\mathbb{C}$-algebra endomorphism, we have $\sigma S(u,v) = S(\sigma u, \sigma v)$ for any u, v in $\mathbb{K}$. Therefore, if $(u,v) \sim^x (u,v')$ then $S(\sigma(u), \sigma(v)) = \sigma(S(u,v)) = \sigma(S(u,v')) = S(\sigma(u), \sigma(v'))$, so $\sigma \cdot (u,v) \sim^x \sigma \cdot (u,v')$. The same argument applies if $(u,v) \sim^y (u',v)$. The general case of $(u,v) \sim^* (u',v')$ follows by induction. □

As a direct corollary, we find the following lemma which ensures the set-wise stability of the orbit under certain endomorphisms of $\mathbb{K}$.

Lemma 3.1.6. *Let $\sigma_x\colon \mathbb{K} \to \mathbb{K}$ be a $k(x)$-algebra endomorphism. Then, for all (u,v) in the orbit, $\sigma_x \cdot (u,v)$ is in the orbit. Similarly, the orbit is also stable under $k(y)$-algebra endomorphisms of $\mathbb{K}$.*

Proof. Let (u,v) be in the orbit, i.e. $(u,v) \sim^* (x,y)$. By Lemma 3.1.5, we find that

$$\sigma_x \cdot (u,v) \sim^* \sigma_x \cdot (x,y) = (x, \sigma_x(y)).$$

By transitivity, we only need to prove that $(x, \sigma_x(y))$ is in the orbit. This is true because $S(x, \sigma_x(y)) = \sigma_x S(x,y) = S(x,y)$ since σ_x fixes $\mathbb{C}(x, S(x,y))$ so $(x, \sigma_x(y)) \sim^x (x,y)$. □

The above two lemmas imply that any $k(x)$ or $k(y)$-algebra automorphism of $\mathbb{K}$ induces a permutation of the vertices of $\mathcal{O}$ which preserves the colored adjacencies, and is therefore a *graph automorphism* of $\mathcal{O}$. The stability result of Lemma 3.1.6 translates as a field theoretic statement.

Theorem 3.1.7. *The extensions $k(\mathcal{O})/k(x), k(\mathcal{O})/k(y)$ and $k(\mathcal{O})/k(x,y)$ are Galois.*

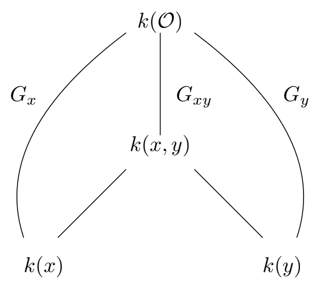


Figure 3.1.2 – The field extensions attached to the orbit

Proof. We first prove that $k(\mathcal{O})/k(x)$ is a Galois extension. Recall that the field extension $k(\mathcal{O})/k(x)$ is algebraic and $\mathbb{K}$ is an algebraic closure of $k(\mathcal{O})$ and $k(x)$. Thus, by Proposition 2.1.7 we only need to prove that $\sigma(k(\mathcal{O})) \subset k(\mathcal{O})$ for every automorphism σ in $\mathrm{Aut}(\mathbb{K}/k(x))$. This follows directly from Lemma 3.1.6. The proof for $k(\mathcal{O})/k(y)$ is entirely symmetric and the field extension $k(\mathcal{O})/k(x,y)$ is Galois as subextension of $k(\mathcal{O})/k(x)$. □

Theorem 3.1.7 gives a Galoisian framework to the orbit, which will be central in our study of Galois invariants and decoupling. Remark that the algebraic extension $k(\mathcal{O})/k(x,y)$ may be of infinite degree. In Figure 3.1.2, we represent the different Galois extensions involved in Theorem 3.1.7 and we denote their Galois groups as

$$G_x = \mathrm{Gal}(k(\mathcal{O})/k(x)) \qquad G_y = \mathrm{Gal}(k(\mathcal{O})/k(y)) \qquad G_{xy} = \mathrm{Gal}(k(\mathcal{O})/k(x,y)).$$

Note that $G_{xy} = G_y \cap G_x$.

3.1.3 The group of the walk and its action on the orbit

In this section, we prove that the orbit $\mathcal{O}$ is the orbit of the pair (x, y) under the action of a certain group that generalizes the one introduced in the small steps case by Bousquet-Mélou and Mishna [BM10, Section 3]. The following definition extends for large steps models the Galoisian construction of [FIM99, Section 2.4] which corresponds to the case of a biquadratic polynomial $\widetilde{K}(x, y)$.

Definition 3.1.8 (Group of the walk)**.** For a model $\mathcal{S}$ with non-univariate step polynomial, we denote by G the subgroup of $\mathrm{Aut}(k(\mathcal{O})/k)$ generated by G_x and G_y, and we call it the *group of the walk*.

As explained in Section 3.1.2, every element of G induces a *graph automorphism* of $\mathcal{O}$, that is, a permutation of the vertices of $\mathcal{O}$ which preserves the colored adjacencies on the orbit $\mathcal{O}$. In Theorem 3.1.12 below, we prove that there exists a finitely generated subgroup of G whose action on $\mathcal{O}$ is faithful and transitive, which is a notable property of the classical group of the walk. It is clear that the group G acts faithfully on the orbit $\mathcal{O}$. Indeed, if an element σ of G is the identity on any element of the orbit then σ is the identity on $k(\mathcal{O})$. Therefore, σ is the identity. The construction of a finitely generated subgroup of G with a transitive action on the orbit requires a bit more work. We first prove two lemmas on the polynomial $\widetilde{K}(x,y)$.

Lemma 3.1.9. *The kernel polynomial $\widetilde{K}(x,y)$ is irreducible in $\mathbb{C}[x,y,t]$. Therefore, it is irreducible as a polynomial in $\mathbb{C}(x,t)[y], \mathbb{C}(y,t)[x]$ and $\mathbb{C}(t)[x,y]$.*

Proof. The kernel polynomial is a degree 1 polynomial in t, therefore it is irreducible in $\mathbb{C}(x,y)[t]$. Moreover, its content is one by construction. Therefore, by Gauss Lemma [Lan02, chap. V par. 6 Theorem 10], the kernel polynomial is irreducible in $\mathbb{C}[x,y][t] = \mathbb{C}[x,y,t]$. Since $S(x,y)$ is not univariate by assumption 3.1.1, the polynomial $\widetilde{K}(x,y)$ does not belong to $\mathbb{C}[x,t]$, so Gauss Lemma asserts that $\widetilde{K}(x,y)$ being irreducible in $\mathbb{C}[x,t][y]$ is also irreducible in $\mathbb{C}(x,t)[y]$. The same reasoning holds for the irreducibility of $\widetilde{K}(x,y)$ in $\mathbb{C}(y,t)[x]$. It is clear that since $\widetilde{K}(x,y)$ is irreducible in $\mathbb{C}[x,y,t]$ and not in $\mathbb{C}(t)$, it is irreducible in $\mathbb{C}(t)[x,y]$. □

Lemma 3.1.10. *The polynomials $\widetilde{K}(Z,y,1/S(x,y))$ and $\widetilde{K}(x,Z,1/S(x,y))$ are respectively irreducible in $k(y)[Z]$ and $k(x)[Z]$.*

Proof. We only prove the first assertion by symmetry of the roles of x and y. Consider the $\mathbb{C}[x]$-algebra homomorphism

$$\begin{aligned}\phi : \mathbb{C}[x,t] &\to \mathbb{C}(x,1/S(x,y)) = k(x)\\ t &\mapsto 1/S(x,y)\end{aligned}$$

Since $S(x,y)$ is not univariate, the fractions x and $1/S(x,y)$ are algebraically independent over $\mathbb{C}$. Therefore the morphism ϕ is one-to-one, so it extends to a field isomorphism $\phi : \mathbb{C}(x,t) \to k(x)$ (onto by definition of $k(x)$), which extends to a $\mathbb{C}$-algebra isomorphism ϕ from $\mathbb{C}(x,t)[y]$ to $k(x)[y]$. Moreover, by Lemma 3.1.9, $\widetilde{K}(x,y)$ is irreducible as a polynomial in $\mathbb{C}(x,t)[y]$. Therefore, since $\widetilde{K}(x,y,1/S(x,y)) = \phi(\widetilde{K}(x,y,t))$ and $\phi(\mathbb{C}(x,t)) = k(x)$, we conclude that the polynomial $\widetilde{K}(x,y,1/S(x,y))$ is irreducible over $k(x)$. □

For large steps models, the extensions $k(\mathcal{O})/k(x)$ and $k(\mathcal{O})/k(y)$ might be of infinite degree, hence the groups G_x and G_y might not be finite, not even finitely generated (unlike the small steps case where they are always cyclic of order 2). However, note that G_{xy} is the stabilizer of the pair (x,y) in the orbit. Therefore, the action of G on (x,y) factors through the left quotients G_x/G_{xy} and G_y/G_{xy} which are proved to be finite in the following lemma.

Lemma 3.1.11. *The group G_{xy} has finite index in G_x and in G_y, with $[G_x : G_{xy}] = \deg_x \widetilde{K}(x,y)$ and $[G_y : G_{xy}] = \deg_y \widetilde{K}(x,y)$.*

Proof. The orbit Ω of y under the action of G_x is a subset of the roots of the polynomial $\widetilde{K}(x,Z,1/S(x,y)) \in k(x)[Z]$. This polynomial is irreducible by Lemma 3.1.10, so G_x acts transitively on its roots by Lemma 2.1.11, hence Ω coincides with the set of roots of $\widetilde{K}(x,Z,1/S(x,y))$ which is a finite set of cardinal $\deg_y \widetilde{K}(x,y)$. Moreover, the stabilizer of y for this action is precisely the group G_{xy}. Therefore, the quotient G_x/G_{xy} can be identified with Ω, which proves that G_{xy} has finite index in G_x with $[G_x : G_{xy}]$. The proof for the subgroup G_y is analogous. $\square$

Therefore, we fix, once and for all, a set $I_x = \{\mathrm{id}, \iota_1^x, \dots, \iota_{m_y+M_y}^x\}$ of representatives of the left cosets of G_x/G_{xy}, and a set $I_y = \{\mathrm{id}, \iota_1^y, \dots, \iota_{m_x+M_x}^y\}$ of representatives of the left cosets of G_y/G_{xy}. By construction, one has

$$G_x = \langle I_x, G_{xy} \rangle, \qquad G_y = \langle I_y, G_{xy} \rangle, \qquad \text{and } G = \langle I_x, I_y, G_{xy} \rangle.$$

We now have all the ingredients to prove the transitivity of the action of a finitely generated subgroup of G on $\mathcal{O}$. We only recall that the distance between two vertices of a graph is the number of edges in a shortest path connecting them.

Theorem 3.1.12 (Transitivity of the action)**.** *The subgroup of G generated by I_x and I_y acts transitively on the orbit $\mathcal{O}$.*

Proof. We show that for all pairs (u, v) of $\mathcal{O}$ there exists an element σ in $\langle I_x, I_y \rangle$ such that $\sigma \cdot (x, y) = (u, v)$. As the graph of the orbit is connected, the proof is done by induction on the distance between (x, y) and (u, v). If (u, v) is at distance zero to (x, y) then $(u, v) = (x, y)$ and we set $\sigma = \mathrm{id}$.

Let (u, v) be in $\mathcal{O}$ of positive distance d to (x, y). Then there exists a pair (u', v') at distance $d - 1$ to (x, y) that is adjacent to (u, v). Without loss of generality, one can assume that (u', v') is x-adjacent to (u, v), that is, $u = u'$. By induction hypothesis, there exists σ in $\langle I_x, I_y \rangle$ such that $\sigma \cdot (x, y) = (u, v')$. Therefore, since $(u, v') \sim^x (u, v)$, the application of σ^{-1} implies by Lemma 3.1.5 that $(x, y) \sim^x (x, \sigma^{-1}(v))$.

Thus, one has $S(x, y) = S(x, \sigma^{-1}v)$, hence both y and $\sigma^{-1}v$ are roots of $\widetilde{K}(x, Z, 1/S(x, y))$ which is an irreducible polynomial in $k(x)[Z]$ by Lemma 3.1.10. Therefore, by Lemma 2.1.11, there is an element σ_x in G_x such that $\sigma_x(y) = \sigma^{-1}(v)$. Since I_x is by definition a set of representatives of the left cosets of G_x/G_{xy}, there exists ι_i^x in I_x such that $\iota_i^x G_{xy} = \sigma_x G_{xy}$. Thus, $\iota_i^x = \sigma_x \circ \tau$ for some $\tau \in G_{xy}$, and $\iota_i^x \cdot (x, y) = \sigma_x \cdot (x, y) = (x, \sigma^{-1}v)$. We hence obtain the identity

$$(\sigma \iota_i^x) \cdot (x, y) = \sigma \cdot (x, \sigma^{-1}v) = (u, v). \qquad \square$$

This result shows that the orbit $\mathcal{O}$ is actually the orbit of the pair (x, y) under the action of a finitely generated subgroup of G. We deduce several easy corollaries:

Corollary 3.1.13. *Let (u, v) and be a pair in $\mathcal{O}$. Then*

(i) The following extensions are isomorphic:

$$k(x, y)/k \simeq k(u, v)/k$$
$$k(x)/k \simeq k(u)/k$$
$$k(y)/k \simeq k(v)/k$$

(ii) The following groups are isomorphic:

$$G_{xy} \simeq G_{uv} \stackrel{\text{def}}{=} \mathrm{Gal}(k(\mathcal{O})/k(u,v))$$
$$G_x \simeq G_u \stackrel{\text{def}}{=} \mathrm{Gal}(k(\mathcal{O})/k(u))$$
$$G_y \simeq G_v \stackrel{\text{def}}{=} \mathrm{Gal}(k(\mathcal{O})/k(v))$$

Moreover, the group G_u acts transitively on the set of pairs x-adjacent to (u,v), and the group G_v acts transitively on the set of pairs y-adjacent to (u,v).

(iii) The field $k(\mathcal{O})$ is the smallest field in $\mathbb{K}$ that contains $k(x,y)$, and so that both extensions $k(\mathcal{O})/k(x)$ and $k(\mathcal{O})/k(y)$ are Galois.

Proof. Note that from Theorem 3.1.12, there exists σ in G such that $\sigma \cdot (x,y) = (u,v)$ and σ' in G such that $\sigma' \cdot (x,y) = (u',v')$. We can now prove the different points.

For (i), it is easy to see that σ induces a k-algebra isomorphism between $k(x,y)$ and $k(u,v)$, whose restriction to $k(u)$ and $k(v)$ also gives isomorphisms.

The proof of the first part of (ii) is similar, the different isomorphisms being given through the conjugation by σ in G. For the second part, since G acts transitively on the set of pairs x-adjacent to (u,v), then so does G_u. Indeed, if $\sigma \cdot (u,v) = (u,v')$, then $\sigma(u) = u$ so $\sigma \in G_u$. The same proof shows that G_v acts transitively on the pairs y-adjacent to (u,v).

We finally prove (iii). We first make the following observation. By Proposition 2.1.3, since the extensions $\mathbb{K}/k(x)$, $k(\mathcal{O})/k(x)$ and $\mathbb{K}/k(y)$, $k(\mathcal{O})/k(y)$ are Galois, the groups $G_x = \mathrm{Gal}(k(\mathcal{O})/k(x))$ and $G_y = \mathrm{Gal}(k(\mathcal{O})/k(y))$ are the restrictions to $k(\mathcal{O})$ of the groups $\widetilde{G_x} \stackrel{\text{def}}{=} \mathrm{Gal}(\mathbb{K}/k(x))$ and $\widetilde{G_y} \stackrel{\text{def}}{=} \mathrm{Gal}(\mathbb{K}/k(y))$. Therefore, one has by Theorem 3.1.12 that the group $\widetilde{G} \stackrel{\text{def}}{=} \langle \widetilde{G_x}, \widetilde{G_y} \rangle$ acts transitively on the orbit.

Now, let M be such that $k(x,y) \subset M \subset \mathbb{K}$, and assume that the extensions $M/k(x)$ and $M/k(y)$ are Galois. Since $M/k(x)$ is Galois and $\mathbb{K}$ is an algebraic closure of M, then for σ_x in $\widetilde{G_x}$, one has $\sigma_x(M) \subset M$. Similarly, since $M/k(y)$ is Galois, one has for σ_y in $\widetilde{G_y}$ that $\sigma_y(M) \subset M$. By induction, we deduce that for all σ in $\widetilde{G}$, one has $\sigma(M) \subset M$.

We now prove that $k(\mathcal{O}) \subset M$, and to do this we only need to show that $k(u,v)$ is contained in M for all pairs (u,v) in the orbit. Since, $\widetilde{G}$ acts transitively on the orbit, let σ be in $\widetilde{G}$ such that $k(u,v) = \sigma(k(x,y))$. From the facts that $\sigma(M) \subset M$ and that $k(x,y) \subset M$, we deduce that $k(u,v) \subset M$. □

3.1.4 Computations of some groups

For large steps models with an infinite orbit, it might be quite difficult to give a precise description of the automorphisms in I_x and I_y. Indeed, they act as a permutation on the infinite orbit $\mathcal{O}$ and their action on x or y is not in general given by a rational fraction in x and y as in the small steps case. When the steps are small or when the orbit is finite, one might be able to give a more precise description of these generators and of the overall group. We give three examples.

Small steps models

For small steps models, we have $k(\mathcal{O}) = k(x,y) = \mathbb{C}(x,y)$ (see Example 3.1.3). Moreover, the field extensions $k(\mathcal{O})/k(x)$ and $k(\mathcal{O})/k(y)$ are both of degree 2. Indeed, the model has small steps, and one step in each direction, so that $\deg_x \widetilde{K}(x,y) = \deg_y \widetilde{K}(x,y) = 2$, so that G_x and G_y are groups of order 2 and thus isomorphic to $\mathbb{Z}/2\mathbb{Z}$. This is similar to the situation of 2.1.9, which was used to define the classical group of the walk.

We may be more precise, and give the explicit generators in terms of the formal group of the walk of 2.2.1. Consider the endomorphisms ϕ and ψ of $\mathbb{C}(x,y)$ defined as follows: for $f(x,y) \in \mathbb{C}(x,y)$, we set $\phi(f) = f(\Phi(x,y))$ and $\psi(f) = f(\Psi(x,y))$. It is easily seen that $\psi \in G_x$, that $\phi \in G_y$, and that they both are non-trivial involutions. Thus, we have

$$G_x = \langle \psi \rangle \simeq \mathbb{Z}/2\mathbb{Z} \qquad G_y = \langle \phi \rangle \simeq \mathbb{Z}/2\mathbb{Z} \qquad G_{xy} = 1$$

Thus, one can choose $I_x = \{\mathrm{id}, \psi\}$ and $I_y = \{\mathrm{id}, \phi\}$.

Note that G is isomorphic to the groups corresponding to both approaches defined in Section 2.1 and Section 2.2.

The model $\mathcal{G}_\lambda$

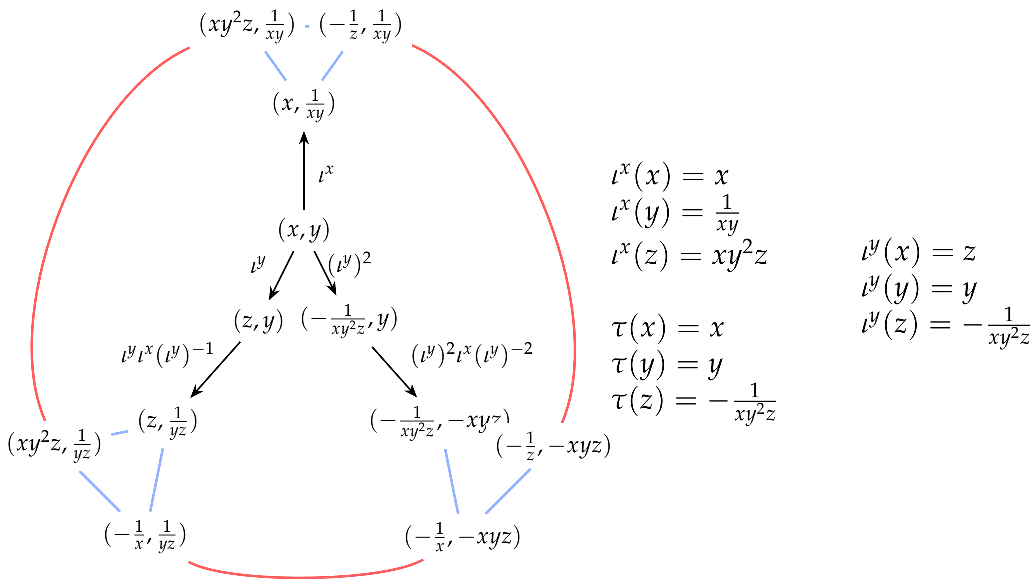


Figure 3.1.3 – The elements of I_x and I_y for the model $\mathcal{G}_\lambda$

In the case of $\mathcal{G}_\lambda$, we saw in Example 3.1.4 that $k(\mathcal{O}) = \mathbb{C}(x,y,z)$ where z is a root of the irreducible polynomial

$$P(Z) = xy^2Z^2 + (x^2y^2 + \lambda xy + x)Z - 1.$$

The extension $k(\mathcal{O})/k(x,y) = k(x,y,z)/k(x,y)$ has degree 2. Thus, we find that

$$G_{xy} \simeq \mathbb{Z}/2\mathbb{Z}.$$

The extension $k(\mathcal{O})/k(y)$ has degree 6, hence its Galois group is either S_3 or $\mathbb{Z}/6\mathbb{Z}$. In this last case, the group G_{xy} would be a normal subgroup of G_y. As $k(x,y) = k(\mathcal{O})^{G_{xy}}$, the extension $k(x,y)/k(y)$ would be Galois by [Sza09, Theorem 1.2.5]. This is impossible since the root z of $\widetilde{K}(Z,y,1/S(x,y))$ is not in $k(x,y)$. Thus, we find that

$$G_y \simeq S_3.$$

The extension $k(\mathcal{O})/k(x)$ has degree 4. Its Galois group has order four and therefore either isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ or to $\mathbb{Z}/4\mathbb{Z}$. Consider $\sigma \in G_x$. If $\sigma \in G_{xy}$, then since G_{xy} has order 2, then $\sigma^2 = 1$. Otherwise, $\sigma = \iota_1^x \tau$ for τ in G_{xy} since $[G : G_{xy}] = 2$. One has $\iota_1^x \cdot (x,y) = (x, \frac{1}{xy})$. Now, since $(x,y) \sim^x (z,y)$, one has by Lemma 3.1.5 that $(x, \frac{1}{xy}) \sim^x (\sigma(z), \frac{1}{xy})$. Therefore, following Figure 3.1.3, we see that $\iota_1^x(z)$ belongs to $\{xy^2z, -\frac{1}{z}\}$. Since, $k(\mathcal{O}) = \mathbb{C}(x,y,z)$, this determines completely ι_1^x on $k(\mathcal{O})$. For both expressions of ι_1^x on $\mathbb{C}(x,y,z)$, we find that $(\iota_1^x)^2(z) = z$. This implies that $(\iota_1^x)^2$ is the identity on $k(\mathcal{O})$. Therefore, $\sigma^2 = (\iota_1^x)^2\tau^2 = 1$. Hence, all elements of G_x have order 2, so that

$$G_y \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

To summarize, we have

$$G_x \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \qquad G_y \simeq S_3 \qquad G_{xy} \simeq \mathbb{Z}/2\mathbb{Z}.$$

for the respective Galois groups, and the above discussion allows to give expressions for the generators in I_x, I_y and G_{xy} in Figure 3.1.3. They satisfy the relations $(\iota^x)^2 = (\iota^y)^3 = \tau^2 = \mathrm{id}$.

Hadamard models

The notion of *Hadamard* models has been introduced by Bostan, Bousquet-Mélou and Melczer who proved that these models are always D-finite [BBM21, Proposition 21]. Hadamard models are characterized by the shape of their Laurent polynomial:

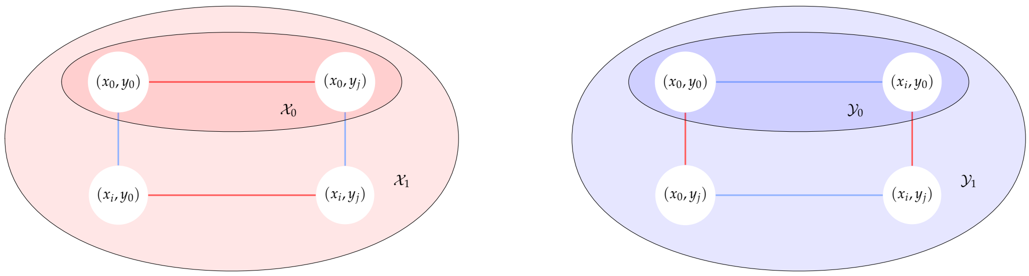
$$S(x,y) = P(x)Q(y) + R(x)$$

for P, Q and R three Laurent polynomials.

The Hadamard models form an interesting class because their orbit is always finite and in the form of a cartesian product. Indeed, we have the following proposition:

Proposition 3.1.14 (Proposition 3.22 in [BBM21])**.** *The orbit of a Hadamard model given by $S(x,y) = P(x)Q(y) + R(x)$ has the form $\mathbf{x} \times \mathbf{y}$ where $\mathbf{x} = x_0, \ldots, x_{m-1}$ the m distinct solutions x_i of $P(X)Q(y) + R(X) = P(x)Q(y) + R(x)$ and $\mathbf{y} = y_0, \ldots, y_{n-1}$ the n distinct solutions y_i of $Q(Y) = Q(y)$. Hence, the field $k(\mathcal{O})$ is equal to $\mathbb{C}(\mathbf{x}, \mathbf{y})$.*

As a consequence, the orbits of the Hadamard models, even though their size might be arbitrarily large, always have diameter two. This means that the distance between any two vertices is at most two as illustrated below:



Our goal in the remaining of this section is to give an explicit description of the group of the walk G for a Hadamard model when the step polynomial has the form $S(x,y) = Q(y) + R(x)$ or $P(x)Q(y)$. In that situation, we shall prove that the group of the walk is a direct product of two simple Galois groups.

Proposition 3.1.15. *Consider a Hadamard model with step polynomial of the form $Q(y) + P(x)$ or $P(x)Q(y)$. The following holds.*

- *The field $k_{inv} = k(x) \cap k(y)$ is $\mathbb{C}(P(x), Q(y))$.*
- *In the notation of Proposition 3.1.14, the elements of $\mathbf{x}$ satisfy $P(x_i) = P(x)$ and the field extensions $\mathbb{C}(\mathbf{x})/\mathbb{C}(P(x))$ and $\mathbb{C}(\mathbf{y})/\mathbb{C}(Q(y))$ are both Galois. We denote their respective Galois groups H_x and H_y.*
- *The group of the walk G is isomorphic to $H_x \times H_y$.*

Before proving Proposition 3.1.15, we recall some terminology. We say that two field extensions L/K and M/K, subfields of a common field Ω, are *algebraically independent* if any finite set of elements of L, that are algebraically independent over K, remains algebraically independent over M.

We say that L/K and M/K are *linearly disjoint* over K if any finite set of elements of L, that are K-linearly independent, are linearly independent over M. The *field compositum* of L and M is the smallest subfield of Ω that contains L and M.

Finally, we say that L/K is a regular field extension if K is relatively algebraically closed in L and L/K is separable. We recall that K is relatively algebraically closed in L if any element of L that is algebraic over K belongs to K. Note that in our setting, all fields are in characteristic zero so L/K is always separable.

Proof. The proof of the first item is obvious. We thus only write the proof of the second one. First, let us prove that $\mathbb{C}(\mathbf{x}, Q(y))/\mathbb{C}(P(x), Q(y))$ is Galois with Galois group isomorphic to H_x. We remark that since x and y are algebraically independent over $\mathbb{C}$, the field extension $\mathbb{C}(P(x), Q(y))/\mathbb{C}(P(x))$ is purely transcendental of transcendence degree one, hence regular. Since $\mathbb{C}(\mathbf{x})/\mathbb{C}(P(x))$ is an algebraic extension, the element $Q(y)$ remains transcendental over $\mathbb{C}(\mathbf{x})$. Thus, the field extensions $\mathbb{C}(\mathbf{x})$ and $\mathbb{C}(P(x), Q(y))$ are algebraically independent over $\mathbb{C}(P(x))$. Thus, by Lemma 2.6.7 in [FJ23], the fields $\mathbb{C}(\mathbf{x})$ and $\mathbb{C}(P(x), Q(y))$ are linearly disjoint over $\mathbb{C}(P(x))$. Then, the field $\mathbb{C}(\mathbf{x}, Q(y))$ that is the compositum of $\mathbb{C}(\mathbf{x})$ and $\mathbb{C}(P(x), Q(y))$, is Galois over $\mathbb{C}(P(x), Q(y))$ with Galois group isomorphic to H_x (see page 35 in [FJ23]). Analogously, one can prove that $\mathbb{C}(\mathbf{y}, P(x))/\mathbb{C}(P(x), Q(y))$ is Galois with Galois group isomorphic to H_y. For convenience, we introduce for the remaining of the proof

$$H'_x = \mathrm{Gal}(\mathbb{C}(\mathbf{x}, Q(y))/\mathbb{C}(P(x), Q(y))) \simeq H_x$$
$$H'_y = \mathrm{Gal}(\mathbb{C}(\mathbf{x}, P(x))/\mathbb{C}(P(x), Q(y))) \simeq H_y.$$

We note that the field extension $\mathbb{C}(\mathbf{x})/\mathbb{C}$ is regular of transcendence degree 1. Since x is transcendental over $\mathbb{C}(\mathbf{y})$, the fields extensions $\mathbb{C}(\mathbf{x})$ and $\mathbb{C}(\mathbf{y})$ are algebraically independent over $\mathbb{C}$ and therefore linearly disjoint over $\mathbb{C}$ by Lemma 2.6.7 in [FJ23]. By the tower property of the linear disjointness (Lemma 2.5.3 in [FJ23]), we find that $\mathbb{C}(\mathbf{x}, Q(y))$ is linearly disjoint from $\mathbb{C}(\mathbf{y})$ over $\mathbb{C}(Q(y))$. Using once again the tower property, we conclude that $\mathbb{C}(\mathbf{x}, Q(y))$ and $\mathbb{C}(\mathbf{y}, P(x))$ are linearly disjoint over

$$k_{\mathrm{inv}} = \mathbb{C}(P(x), Q(y)) = k(x) \cap k(y).$$

Thus, Lemma 2.5.6 in [FJ23] implies that the following restriction map is a group isomorphism.

$$\begin{aligned} G &\longrightarrow H'_x \times H'_y \\ \sigma &\longmapsto (\sigma|_{\mathbb{C}(\mathbf{x}, Q(y))}, \sigma|_{\mathbb{C}(\mathbf{y}, P(x))}). \end{aligned}$$

By the above, we conclude that G is isomorphic to $H_x \times H_y$. □

3.1.5 Orbit sums

One of the purposes of the orbit is to provide a nice family of changes of variables, in the sense that the kernel $K(x,y) = 1 - t\,S(x,y)$ is constant on the orbit: for all pairs (u,v) of the orbit, $K(u,v) = K(x,y)$ (because $S(x,y) = S(u,v)$). This polynomial being a factor of the left-hand side of the functional equation satisfied by the generating function, one can evaluate the variables (x,y) at any pair (u,v) of the orbit and obtain what is called an *orbit equation*. Indeed, the generating function $Q(x,y)$ and its sections $Q(x,0)$ and $Q(0,y)$ belong to the ring of formal power series in t with coefficients in $\mathbb{C}[x,y]$ so that their evaluation at (u,v) belongs to the ring $\mathbb{C}[\mathcal{O}][\![t]\!]$. Note that such an evaluation leaves the variable t fixed.

The strategy developed in [BBM21, Section 4] for models with small forward steps consists in forming linear combinations of these orbit equations so that the resulting equation is free from sections. From the section-free equation, Bostan, Bousquet-Mélou and Melczer sometimes succeed in isolating the generating function $Q(x,y)$ and expressing it as a diagonal of algebraic fractions which leads to its D-finiteness by [Lip88].

For models with small backward steps, it is quite easy to produce a section-free equation from

$$\widetilde{K}(x,y)Q(x,y) = xy + A(x) + B(y)$$

when the orbit contains a cycle. However, it is very unlikely that, for models with small backward steps and at least one large step, such a section free equation suffices to characterize the generating function.

In the context of chapters 3 and 4, we want to evaluate the variables (x, y, t) at $(u, v, 1/S(x, y))$ for (u, v) an element of the orbit. Since $\widetilde{K}(u, v, 1/S(x, y)) = 0$ for any element (u, v) of the orbit $\mathcal{O}$, such an evaluation is similar to the kernel method used in [KR12] for models with small steps.

More precisely, let us define a 0-*chain* as a formal $\mathbb{C}$-linear combination of elements of the orbit $\mathcal{O}$ with finite support. This terminology is borrowed from graph homology (see Section 4.4 for some basic introduction). Let $\gamma = \sum_{(u,v)\in\mathcal{O}} c_{(u,v)}(u, v)$ be a zero chain. Since the coefficients $c_{(u,v)}$ are complex and almost all zero, the evaluation P_γ of a polynomial $P(x, y) \in \mathbb{C}[x, y, t]$ at γ is defined as

$$P_\gamma = \sum_{(u,v)\in\mathcal{O}} c_{(u,v)} P(u, v, 1/S(x, y)),$$

and belongs to $k[\mathcal{O}]$. The evaluation of $\widetilde{K}(x, y)$ at any 0-chain vanishes so that one can not evaluate a rational fraction in $\mathbb{C}(x, y, t)$ whose denominator is divisible by $\widetilde{K}(x, y)$. This motivates the following definition.

Definition 3.1.16. Let $H(x, y) = \frac{A(x,y)}{B(x,y)}$ be a rational fraction in $\mathbb{C}(x, y, t)$ where $A(x, y)$ and $B(x, y)$ are relatively prime polynomials in $\mathbb{C}[x, y, t]$. We say that $H(x, y)$ is a *regular fraction* if $B(x, y)$ is not divisible by the kernel polynomial $\widetilde{K}(x, y)$ in $\mathbb{C}[x, y, t]$.

Remark 3.1.17. Since $S(x, y)$ is not univariate, the kernel polynomial involves all three variables x,y and t, so does a multiple of $\widetilde{K}(x, y)$ (by a simple degree argument). Therefore, any fraction in $\mathbb{C}(x, t)$ or $\mathbb{C}(y, t)$ is regular. ■

We endow the set of regular fractions in $\mathbb{C}(x, y, t)$ with the following equivalence relation: two regular fractions $H(x, y)$, and $G(x, y)$ are *equivalent* if there exists a regular fraction $R(x, y)$ such that $H(x, y) - G(x, y) = \widetilde{K}(x, y)R(x, y)$. We denote by $\mathcal{C}$ the set of equivalence classes.

Since the equivalence relation is compatible with the addition and multiplication of fractions, one easily notes that $\mathcal{C}$ can be endowed with a ring structure. Moreover, since $\widetilde{K}(x, y)$ is irreducible in $\mathbb{C}[x, y, t]$, any non-zero class is invertible proving that $\mathcal{C}$ is a field. Indeed, if $H(x, y)$ is a regular fraction that is not equivalent to zero, then one can write $H(x, y) = \frac{A(x,y)}{B(x,y)}$ with $A(x, y), B(x, y) \in \mathbb{C}[x, y, t]$ relatively prime and $\widetilde{K}(x, y)$ does not divide $A(x, y)$ nor $B(x, y)$. Thus, the fraction $\frac{B(x,y)}{A(x,y)}$ is regular and its class in $\mathcal{C}$ is an inverse of the class of $\frac{A(x,y)}{B(x,y)}$.

Moreover, since $S(x, y)$ is not univariate, any non-zero element in $\mathbb{C}(x, t)$ or $\mathbb{C}(y, t)$ is a regular fraction which is not equivalent to zero. Therefore, the fields $\mathbb{C}(x, t)$ and

$\mathbb{C}(y,t)$ embed into $\mathcal{C}$. By an abuse of notation, we denote by $\mathbb{C}(x,t)$ and $\mathbb{C}(y,t)$ their image in $\mathcal{C}$.

Proposition 3.1.18. *For a fraction $H(x,y)$ in $\mathbb{C}(x,y,t)$ and (u,v) in $\mathcal{O}$, the evaluation $H(u,v,1/S(x,y))$ of $H(x,y)$ at (u,v) is a well defined element of $\mathbb{K}$ if and only if $H(x,y)$ is a regular fraction. Moreover, the $\mathbb{C}$-algebra homomorphism*

$$\begin{aligned} \phi : \mathcal{C} &\longrightarrow k(x,y) \\ P(x,y,t) &\longmapsto P(x,y,1/S(x,y)) \end{aligned}$$

is well defined and is a field isomorphism which maps isomorphically $\mathbb{C}(t)$ onto $k = \mathbb{C}(S(x,y))$, $\mathbb{C}(x,t)$ onto $k(x)$ and $\mathbb{C}(y,t)$ onto $k(y)$.

Proof. Recall that by Theorem 3.1.12, given a pair $(u,v) \in \mathcal{O}$, there exists $\sigma \in G$ such that $\sigma \cdot (x,y) = (u,v)$. The automorphism σ induces a k-algebra isomorphism between $k(x,y)$ and $k(u,v)$ so that the evaluation at $(x,y,1/S(x,y))$ composed by σ is the evaluation at $(u,v,1/S(x,y))$. Thus, we only need to prove the first part of the proposition for the evaluation at $(x,y,1/S(x,y))$.

Since $\widetilde{K}(x,y,1/S(x,y)) = 0$, it is clear that one can not evaluate a fraction that is not regular. Thus, we only need to show that the evaluation of a regular fraction at $(x,y,1/S(x,y))$ is well defined. Let us write $H(x,y) = \frac{A(x,y)}{B(x,y)}$ where $A(x,y)$ and $B(x,y)$ are relatively prime in $\mathbb{C}[x,y,t]$, and the kernel polynomial $\widetilde{K}(x,y,t)$ does not divide $B(x,y)$. Since $\widetilde{K}(x,y)$ does not divide $B(x,y)$ in $\mathbb{C}[x,y,t]$ and $\widetilde{K}(x,y)$ has content 1 in $\mathbb{C}[x,y][t]$, the polynomial $\widetilde{K}(x,y)$ does not divide $B(x,y)$ in $\mathbb{C}(x,y)[t]$ (it is a straightforward application of Gauss Lemma). In $\mathbb{C}(x,y)[t]$, the Euclidean division of $B(x,y)$ by $\widetilde{K}(x,y)$, which has degree 1 in t, is therefore of the form

$$B(x,y) = \widetilde{K}(x,y)M(x,y) + R(x,y),$$

where $R(x,y)$ is a non-zero element of $\mathbb{C}(x,y)$. Evaluating this identity at $(x,y,1/S(x,y))$ yields $B(x,y,1/S(x,y)) = R(x,y)$. Since x,y are algebraically independent over $\mathbb{C}$, one finds that $R(x,y) \neq 0$ so that $B(x,y,1/S(x,y))$ is non-zero and $H(x,y,1/S(x,y))$ is well defined.

By Lemma 3.1.9, the kernel polynomial $\widetilde{K}(x,y)$ is irreducible as a polynomial in $\mathbb{C}(t)[x,y]$. The ring $R = \mathbb{C}(t)[x,y]/(\widetilde{K}(x,y))$ is therefore an integral domain. By [Mat80, page 9, (1K)], its quotient field is precisely $\mathcal{C}$. Now, the evaluation map from $\mathbb{C}(t)[x,y]/(\widetilde{K}(x,y))$ to $k[x,y]$ is a ring isomorphism which maps isomorphically $\mathbb{C}(t)$ onto k. The latter ring isomorphism extends to an isomorphism between the quotient field $\mathcal{C}$ of $\mathbb{C}(t)[x,y]/(\widetilde{K}(x,y))$ and the quotient field $k(x,y)$ of $k[x,y]$ which concludes the proof. □

If $H(x,y)$ is a regular fraction, we denote $H_{(u,v)}$ its evaluation at an element (u,v) of the orbit and we can extend this evaluation by $\mathbb{C}$-linearity to any 0-chain γ. We denote by H_γ the corresponding element in $k(\mathcal{O})$. Such an evaluation is called an *orbit sum*.

We let the group G act on 0-chains by $\mathbb{C}$-linearity, that is,

$$\sigma \cdot \left(\sum_{(u,v)\in\mathcal{O}} c_{(u,v)}\,(u,v) \right) \overset{\text{def}}{=} \sum_{(u,v)\in\mathcal{O}} c_{(u,v)}\,\sigma\cdot(u,v).$$

The following lemma shows that the evaluation morphism is compatible with the action of G on $k(\mathcal{O})$ and on 0-chains.

Lemma 3.1.19. *Let σ be an element of G, γ be a 0-chain, and $H(x,y)$ be a regular fraction in $\mathbb{C}(x,y,t)$. Then $\sigma(H_\gamma) = H_{\sigma\cdot\gamma}$.*

Proof. Let (u,v) be an element in the orbit. Since σ fixes $k = \mathbb{C}(S(x,y))$, we have

$$\sigma(H_{(u,v)}) = \sigma\left(H(u,v,1/S(x,y)\right) = H\left(\sigma(u),\sigma(v),1/S(x,y)\right) = H_{\sigma\cdot(u,v)}.$$

The general case follows by $\mathbb{C}$-linearity. □

Two equivalent regular fractions have the same evaluation in $k(\mathcal{O})$. Thus, certain classes of regular fractions can be characterized by the Galoisian properties of their evaluation in $k(\mathcal{O})$. This idea underlies the Galoisian study of invariants and decoupling in Section 3.2 and Chapter 4. To conclude, we want to compare the equivalence relation among regular fractions that are elements of $\mathbb{C}_{\text{mul}}(x,y)((t))$ and the t-equivalence (see Section 2.2.2 for notation).

Proposition 3.1.20. *Let $F(x,y) \in \mathbb{C}_{mul}(x,y)((t))$ that is also a regular fraction in $\mathbb{C}(x,y,t)$. If $F(x,y)$ is t-equivalent to 0, that is, the t-expansion of $F(x,y)/\widetilde{K}(x,y)$ has poles of bounded order at 0, then the fraction $F(x,y)/\widetilde{K}(x,y)$ is regular so that the regular fraction $F(x,y)$ is equivalent to zero by definition.*

Proof. Our proof starts by following the lines of the proof of Lemma 2.6 in [Bou23]. Assume that $F(x)$ is t-equivalent to 0, so that there exists $H(x,y) \in \mathbb{C}_{\text{mul}}(x,y)((t))$ with poles of bounded order at 0 such that

$$F(x,y) = \widetilde{K}(x,y)H(x,y). \tag{3.1.1}$$

Analogous arguments to Lemma 2.6 in [Bou23] show that there exists a root $X(y,t)$ of $K(.,y,t) = 0$ that is a formal power series in t with coefficients in an algebraic closure of $\mathbb{C}(y)$ and with constant term 0. Since $H(x,y)$ and $F(x)$ have poles of bounded order at 0, one can specialize (3.1.1) at $x = X(y,t)$ and find $F(X(y,t),y,t) = 0$. Writing $F = \frac{P}{Q}$ where $P(x,y), Q(x,y) \in \mathbb{C}[x,y,t]$ are relatively prime, one finds that $P(X(y,t),y,t) = 0$. Since $\widetilde{K}(.,y,t)$ is an irreducible polynomial over $\mathbb{C}(y,t)$ by Lemma 3.1.9, we conclude that $\widetilde{K}(x,y)$ divides $P(x,y)$. Because $P(x,y)$ and $Q(x,y)$ are relatively prime, we find that $\widetilde{K}(x,y)$ doesn't divide $Q(x,y)$ which concludes the proof. □

Clearly, the regular fraction $\frac{\widetilde{K}(x,y)}{y-t}$ is equivalent to zero but not t-equivalent to zero, so the converse of Proposition 3.1.20 is false. With the strategy presented in Section 2.2.2

(which is summarized and applied in Chapter 5) in mind, we will use in the next sections the notion of equivalence on regular fractions and its Galoisian interpretation to produce pairs of *Galois invariants* and *Galois decoupling pairs*. For each pair of Galois invariants and decoupling functions constructed for the models presented in Section 4.6, it happens that any equivalence relation among these regular fractions is actually a t-equivalence. Unfortunately, we do not have any theoretical argument yet to explain this phenomenon.

3.2 Galois invariants

In this section, we discuss a form of invariants, called *Galois invariants*, whose interest is to provide rational t-invariants (see 2.2.2). We prove that the finiteness of the orbit is equivalent to the existence of a non-constant pair of such invariants. Moreover, we show that if the orbit is finite, the field of Galois invariants has the form $k(c)$ for some element c transcendental over k, which is easy to obtain from the data of a finite orbit.

3.2.1 Galois formulation of invariants

In Section 2.2.2, we needed to construct t-invariants that were rational fractions, that is, pairs $(I(x), J(y))$ satisfying an equation of the form

$$I(x) - J(y) = \widetilde{K}(x,y)R(x,y)$$

with R having poles of bounded order at zero (I and J are t-equivalent).

We introduce the weaker notion of pair of *Galois invariants* based on rational equivalence. Our definition extends Definition 4.3 in [BBR21] to the large steps context.

Definition 3.2.1. Let $(I(x,t), J(y,t))$ be a pair of rational fractions in $\mathbb{C}(x,t) \times \mathbb{C}(y,t)$ (hence regular, as they are univariate). We say that this pair is a *pair of Galois invariants* if there exists a regular fraction $R(x,y)$ such that $I(x,t) - J(y,t) = \widetilde{K}(x,y)R(x,y)$, that is, the regular fractions $I(x,t)$ and $J(y,t)$ are equivalent.

From Proposition 3.1.20, a pair of rational t-invariants is a pair of Galois invariants. Therefore, it is justified to look for a pair of Galois invariants first, and then to check by hand if their difference is t-equivalent to 0.

Moreover, the notion of pairs of Galois invariants is purely algebraic while the notion of pairs of t-invariants involves some analytic considerations which might be difficult to handle. Using Lemma 3.1.18, the set of pairs of Galois invariants corresponds to a subfield of $k(\mathcal{O})$ which can be easily described.

Proposition 3.2.2. *Let $P = (I(x,t), J(y,t))$ be a pair of fractions in $\mathbb{C}(x,t) \times \mathbb{C}(y,t)$. Then P is a pair of Galois invariants if and only if the evaluations $I_{(x,y)}$ and $J_{(x,y)}$ are equal, and thus belong to $k(x) \cap k(y) \subset k(\mathcal{O})$. Moreover, the pair P is a constant pair of Galois invariants if and only if $I_{(x,y)} = J_{(x,y)}$ is in k.*

Therefore we denote the field $k(x) \cap k(y)$ as k_{inv} and, by an abuse of terminology, call its elements Galois invariants. The definition of the group G and the Galois correspondence applied to $k(\mathcal{O})/k(x)$ and $k(\mathcal{O})/k(y)$ show that f in $k(\mathcal{O})$ is a Galois invariant if and only if f is fixed by G. Moreover, Proposition 3.2.2 reduces the question of the existence of a nonconstant pair of Galois invariants to the question of deciding whether $k_{\text{inv}} = k$ or not.

3.2.2 Existence of nontrivial Galois invariants and finiteness of the orbit

The structure of $k(\mathcal{O})/k(x)$ and $k(\mathcal{O})/k(y)$, both Galois, corresponds in geometry to a *finite algebraic correspondence*. This is studied in a geometric setting in [Fri78]. In particular, Theorem 1 and the lemma following Theorem 1 imply when rephrased in our context that the existence of a non-constant pair of Galois invariants is equivalent to the finiteness of the orbit. These theorems are more general, holding in positive characteristic and in higher dimensional context. In Theorem 3.2.3 below, we rephrase and prove Fried's Theorem in our context. This generalizes [BBR21, Theorem 7] in the small steps case.

Theorem 3.2.3. *The following are equivalent:*

1. *The orbit $\mathcal{O}$ is finite.*
2. *There exists a finite Galois extension M of $k(x)$ and $k(y)$ such that* $\mathrm{Gal}(M/k(x))$ *and* $\mathrm{Gal}(M/k(y))$ *generate a finite group* $\langle \mathrm{Gal}(M/k(x)), \mathrm{Gal}(M/k(y))\rangle$ *of automorphisms of M.*
3. *There exists a nontrivial Galois invariant, that is, $k \subsetneq k_{inv}$.*

Proof. (1) $\Rightarrow$ (2): Set $M = k(\mathcal{O})$. The group $G = \langle G_x, G_y \rangle$ acts faithfully on the orbit, so it embeds as a subgroup of $S(\mathcal{O})$, the group of permutations of the pairs of the orbit $\mathcal{O}$. The orbit is finite, therefore G is finite.

(2) $\Rightarrow$ (3): Write $H = \langle \mathrm{Gal}(M/k(x)), \mathrm{Gal}(M/k(y))\rangle$. By the same argument as in the beginning of Section 3.1.3, the field M^H is the field k_{inv} of Galois invariants. Since H is finite, the extension M/k_{inv} is finite of degree $|H|$, hence the subextension $k(x)/k_{\text{inv}}$ is also finite. Since the extension $k(x)/k$ is transcendental by hypothesis on $\mathcal{S}$, we conclude that $k \subsetneq k_{\text{inv}}$. Proposition 3.2.2 yields the existence of a pair of nontrivial Galois invariants.

(3) $\Rightarrow$ (1): Let $(I(x,t), J(y,t))$ be a pair of nontrivial Galois invariants. By the assumption on the model, $S(x,y)$ and x are algebraically independent over $\mathbb{C}$. Since $I(x, 1/S(x,y))$ is not in $\mathbb{C}(1/S(x,y))$ by Lemma 3.1.18, this implies that the extension $k(I(x, 1/S(x,y)))/k$ is transcendental. As the transcendence degree of $k(x)$ over k is 1, this implies that the extension $k(x)/k(I(x, 1/S(x,y)))$ is algebraic, hence x is algebraic over k_{inv}, with minimal polynomial $P(x)$.

The group G leaves k_{inv} fixed. Thus the orbit of x in $\mathbb{K}$ under the action of G is a subset of the roots of $P(x)$. By Theorem 3.1.12, the action of G is transitive on the orbit, hence the set $G \cdot x = \{u \in \mathbb{K} : \exists \sigma \in G, u = \sigma x\} = \{u \in \mathbb{K} : \exists v \in \mathbb{K}, (u,v) \in \mathcal{O}\}$

is finite. As there are $\deg_y \widetilde{K}(x,y)$ pairs of the orbit with first coordinate u for each u in $G \cdot x$, we conclude that $\mathcal{O}$ is finite. □

Example 3.2.4 (Hadamard models)**.** Consider a Hadamard model (see the corresponding section in 3.1.4) given by the polynomial

$$S(x,y) = P(x)Q(y) + R(x).$$

Then it is straightforward to see that the pair $\left(\frac{t^{-1}-R(x)}{P(x)}, Q(y)\right)$ is a pair of nontrivial Galois invariants. Hence the orbit of a Hadamard model is always finite by Theorem 3.2.3, thus giving an alternative proof to the finiteness of the orbit. ■

Note that Theorem 3.2.3 implies that the extension $k(\mathcal{O})/k_{\text{inv}}$ is finite and Galois, having Galois group $G = \langle G_x, G_y \rangle$.

3.2.3 Effective construction

In order to apply the algebraic strategy presented in Section 2.2.2, we want to find explicit nonconstant rational t-invariants. As already mentioned, we shall first construct explicitly the field of Galois invariants and then search among these Galois invariants the potential rational t-invariants.

In the small steps case, an orbit sum argument was used to construct a pair of Galois invariants [BBR21, Theorem 4.6]. This construction generalizes mutatis mutandis to the large steps case, and is reproduced here to show one way to exploit the group of the walk.

Lemma 3.2.5. *Let ω be the 0-chain $\frac{1}{|\mathcal{O}|}\sum_{a\in\mathcal{O}} a$. Then, for any regular fraction $H \in \mathbb{Q}(x,y,t)$ the element H_ω is a Galois invariant.*

Proof. Let $H(x,y)$ be a regular fraction. Since, by Theorem 3.1.12, the group G acts faithfully on $\mathcal{O}$, the 0-chain ω is invariant by the action of G. Thus, by Lemma 3.1.19, for all σ in G, $\sigma(H_\omega) = H_{\sigma\cdot\omega} = H_\omega$. Therefore, by the Galois correspondence, H_ω is a Galois invariant. □

Unfortunately, a non-constant regular fraction $H(x,y)$ might have a constant evaluation, that is, H_ω might belong to k. Indeed, for the step polynomial $S(x,y) = (x + \frac{1}{x})(y + \frac{1}{x})$ and the regular fraction $H(x,y) = xy$, the element $H_\omega = S(x,y)$ belongs to $k = \mathbb{Q}(S(x,y))$ but $H_{(x,y)} = xy$ is non-constant. Thus, one has to choose carefully $H(x,y)$ in order to avoid this situation which is precisely the strategy used in [BBR21, Theorem 4.6]. Below, we describe an alternative construction which is easier to compute effectively and yields a complete description of the field k_{inv}.

Consider first this simple observation. Since x is algebraic over k_{inv}, we can consider its minimal polynomial $\mu_x(Z)$ in $k_{\text{inv}}[Z]$. One of its coefficients must be in $k_{\text{inv}} \setminus k$ because x is transcendental over k. Thus, such a coefficient is a non-trivial Galois invariant.

A more sophisticated argument using a constructive version of Lüroth's Theorem says actually much more about such a coefficient.

Theorem 3.2.6 (Lüroth's Theorem [Rot15], Th. 6.66)**.** *Let $k(x)$ be a field with x transcendental over k and $k \subset L \subset k(x)$ a subfield. If x is algebraic over K, then any coefficient c of its minimal polynomial $\mu_x(Z)$ over L that is not in k is such that $L = k(c)$.*

Applying this result to the tower $k \subset k_{\text{inv}} \subset k(x)$, not only can we find nontrivial Galois invariants among the coefficients of μ_x, but any one of them generates the field of Galois invariants. In one sense, these coefficients contain all the information on the Galois invariants attached to the model. Therefore, all that remains is to compute effectively the polynomial $\mu_x(Z)$.

By irreducibility of the polynomial $\mu_x(Z)$ in $k_{\text{inv}}[Z]$, the Galois group $G = \text{Gal}(k(\mathcal{O})/k_{\text{inv}})$ acts transitively on its roots. By Theorem 3.1.12, the orbit of x under the action of G is the set of left coordinates of the orbit. Therefore, $\mu_x(Z)$ is precisely the vanishing polynomial of the left coordinates of the orbit, which is exactly computed in the construction of the orbit in [BBM21, Section 3.2]. We detail this construction in Section 4.5.2.

In order to find an explicit pair of non-constant Galois invariants $(I(x), J(y))$, we compute $\mu_x(Z)$. Each of its coefficients lies in k_{inv} and corresponds via Proposition 3.1.18 to a pair of Galois invariants: the non-constant coefficients of $\mu_x(Z)$ leading to nontrivial Galois invariants. One can therefore lift $\mu_x(Z)$ either as a polynomial in $\mathbb{C}(x,t)[Z]$ or as a polynomial in $\mathbb{C}(y,t)[Z]$. The lifts of the polynomial $\mu_x(Z)$ to $\mathbb{C}(x,t)[Z]$ and to $\mathbb{C}(y,t)[Z]$ can be computed directly when constructing the orbit, see 4.5.2, and their difference is a multiple of $\widetilde{K}(x,y)$ by a polynomial in Z whose coefficients are regular fractions.

Example 3.2.7 (Hadamard models)**.** We saw in Example 3.2.4 that the field of Galois invariants of a Hadamard model is nontrivial. We may be more precise. For a Hadamard model given by the polynomial $S(x,y) = P(x)Q(y) + R(x)$, then k_{inv} coincides with $k(Q(y))$.

Indeed, writing $Q(y) = A(y)/B(y)$ with A and B relatively prime, we know that the right coordinates of the orbit are the roots of the polynomial

$$\mu_y(Y) = B(Y) - A(Y)Q(y) \in k(Q(y))[Y] \subset k_{\text{inv}}[y].$$

Thus, the coefficients of this polynomial generate the field of Galois invariants, implying that $k(Q(y)) \subset k_{\text{inv}} \subset k(Q(y))$, which shows the claim. ∎

Example 3.2.8 (The model $\mathcal{G}_\lambda$)**.** Consider the model $\mathcal{G}_\lambda$. Its orbit type is $\mathcal{O}_{12}$. We compute the lift of $\mu_x(Z)$ in $\mathbb{C}(x,t)[Z]$ as

$$Z^6 \boxed{-\frac{(\lambda^2 x^3 + x^6 + x^4 - x^2 - 1)\,t^2 + x^2\lambda\,(x^2-1)\,t - x^3}{t^2 x\,(x^2+1)^2}} Z^5 + \frac{t+\lambda}{t} Z^4$$
$$-2\frac{x^6 t^2 + \left(-\frac{\lambda^2 t^2}{2} + \frac{1}{2}\right) x^5 + t\,(t+\lambda)\,x^4 + (-t^2 - \lambda\,t)\,x^2 - \frac{(\lambda^2 t^2 - 1)x}{2} - t^2}{t^2 x\,(x^2+1)^2} Z^3$$
$$-\frac{(t+\lambda)\,Z^2}{t} - \frac{(\lambda^2 x^3 + x^6 + x^4 - x^2 - 1)\,t^2 + x^2\lambda\,(x^2-1)\,t - x^3}{t^2 x\,(x^2+1)^2} Z - 1$$

and in $\mathbb{C}(y,t)[Z]$ as

$$Z^6 + \boxed{\frac{-t\,y^4 + \lambda t y + y^3 + t}{t\,y^2}} Z^5 + \frac{t+\lambda}{t} Z^4 - 2\frac{\left(y^4 - \frac{1}{2}y^2\lambda^2 - y\lambda - 1\right)t^2 - t\,y^3 + \frac{y^2}{2}}{t^2y^2} Z^3$$
$$- \frac{(t+\lambda)}{t} Z^2 + \frac{\left(-t\,y^4 + \lambda t y + y^3 + t\right)}{t\,y^2} Z - 1.$$

The non-constant coefficients of $\mu_x(Z)$ are the coefficients a_5,a_3,a_1 of Z^5,Z^3 and Z. One sees easily that $a_5 = a_1$ and that $2a_5 = a_3 + \frac{-\lambda^2 t^2 + 1}{t^2}$. The coefficient a_5 yields the following pair of non-trivial Galois invariants $(I(x), J(y))$

$$\left(-\frac{\left(\lambda^2\,x^3 + x^6 + x^4 - x^2 - 1\right)t^2 + x^2\lambda\left(x^2-1\right)t - x^3}{t^2 x\left(x^2+1\right)^2}, \frac{-t\,y^4 + \lambda t y + y^3 + t}{t\,y^2} \right).$$

We check that $\frac{I(x)-J(x)}{\tilde{K}(x,y)}$ has poles of bounded order at 0, hence $(I(x), J(y))$ is a pair of t-invariants. Moreover, Theorem 3.2.6 says that $k_{\text{inv}} = k\left(I(x, 1/S(x,y))\right)$, so any pair of Galois invariants for $\mathcal{G}_\lambda$ is a fraction in the pair $(I(x), J(y))$. ∎

Example 3.2.9. The orbit type of the model with step polynomial $S(x,y) = x + \frac{x}{y} + \frac{y}{x^2} + \frac{1}{x^2}$ is $\mathcal{O}_{18}$ (see Figure 3.1.1). With our method, we find the following pair of Galois invariants

$$\left(\frac{\left(-x^9 - 3x^3 + 1\right)t^2 + \left(x^8 + x^5 - 2x^2\right)t + x^4}{x^6 t^2}, \frac{\left(y^3 + 3y + 1\right)(y+1)^3 t^3 + y^4}{y^2 t^3 (y+1)^3} \right).$$

One can also check by looking at the t-expansions that it is a pair of t-invariants. ∎

Chapter 4

Decoupling with a finite group

In Section 2.2.2, a strategy for showing algebraicity of some models with small steps has been presented, relying on the construction of invariants. Notably, writing

$$xy \equiv F(x) + G(y)$$

for some fractions $F(x)$ and $G(y)$ (a t-decoupling, see Definition 2.2.8) was crucial in order to construct the pair of invariants involving the sections $Q(x,0)$ and $Q(0,y)$. One could imagine to perform the same proof for walks having a more general starting point (a,b), which in turn would require to compute the t-decoupling of the fraction $x^{a+1}y^{b+1}$. Hence, this justifies to study the problem of decoupling an arbitrary fraction $H(x,y)$.

For weighted models with small steps, this problem was well studied. The first traces appear in [FIM99, Section 4]. When the classical group of the walk is finite (corresponding to the case where the orbit is cyclic), [FIM99, Theorem 4.2.9 and Theorem 4.2.10] give Galoisian conditions to guarantee the existence of rational solutions to the equation (1.2.3). These conditions also appear in [BBR21] in the context of a finite formal group of the walk. As seen in Section 2.2.2, the fact that a regular fraction $H(x,y)$ admits a decoupling is completely characterized by the fact that its alternating orbit sum is zero, by [BBR21, Theorem 4.11]. Moreover, the same theorem yields that the fractions $F(x)$ and $G(y)$ that satisfy $H(x,y) = F(x) + G(y) \mod \widetilde{K}(x,y)$ can be explicitly computed in terms of $H(x,y)$ also using an orbit sum.

In this chapter, we extend this criterion to the case of an arbitrary finite orbit using the terminology of Chapter 3. Like we did in the previous chapter to study rational invariants, we give a notion of decoupling in the context of $k(\mathcal{O})$, called *Galois decoupling*. Then, using the interaction of the group with the orbit devised earlier, we give a necessary and sufficient condition for the Galois decoupling of a regular fraction $H(x,y)$, so that it admits a Galois decoupling if and only if $H_\alpha = 0$ for some 0-chain α of the orbit. Moreover, the solutions $F(x)$ and $G(y)$ of the decoupling problem are also expressed as explicit orbit sums of $H(x,y)$, with $F(x) = H_{\widetilde{\gamma}_x}$ and $G(y) = H_{\widetilde{\gamma}_y}$ for some 0-chains $\widetilde{\gamma}_x$ and $\widetilde{\gamma}_y$. As in the cyclic case, the 0-chains α, $\widetilde{\gamma}_x$ and $\widetilde{\gamma}_y$ only depend on the orbit type. We also address the problem of effective computations. We show in Section 4.5.1 that under some mild assumptions on the orbit type, these orbit sums admit an expression suitable for computer algebra. The construction is illustrated in Section 4.6 for some finite orbit types.

4.1 Galois formulation of decoupling

As in the previous section, we adapt the notion of decoupling introduced in Section 2.2.2 to our Galoisian framework. The definition below is the straightforward analogue of Definition 4.7 in [BBR21] for large steps models.

Definition 4.1.1 (Galois decoupling of a regular fraction)**.** Let $H(x,y)$ be a regular fraction in $\mathbb{C}(x,y,t)$. A pair of fractions $(F(x),G(y))$ in $\mathbb{C}(x,t)\times\mathbb{C}(y,t)$ is called a *Galois decoupling pair* for the fraction $H(x,y)$ if there exists a regular fraction $R(x,y)$ satisfying

$$H(x,y) = F(x) + G(y) + \widetilde{K}(x,y)R(x,y).$$

We call such an identity a *Galois decoupling* of the fraction $H(x,y)$.

Thanks to Proposition 3.1.20, if a regular fraction admits a decoupling with respect to the t-equivalence then it admits a Galois decoupling. Analogously to the notion of Galois invariants and as a corollary of Proposition 3.1.18, one can interpret the Galois decoupling as an identity in the extension $k(\mathcal{O})$.

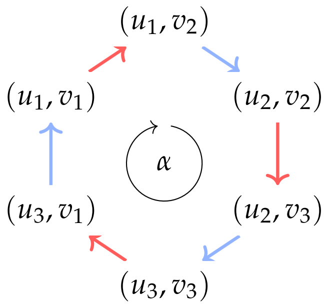
Proposition 4.1.2. *Let $H(x,y)$ be a regular fraction in $\mathbb{C}(x,y,t)$. Then $H(x,y)$ admits a Galois decoupling if and only if $H_{(x,y)}$ can be written as $f+g$ with f in $k(x)$ and g in $k(y)$.*

By an abuse of terminology, we call any identity $H_{(x,y)} = f+g$ with f in $k(x)$ and g in $k(y)$ a Galois decoupling of $H(x,y)$. Furthermore, these last two conditions can be reformulated algebraically via the Galois correspondence applied to the extensions $k(\mathcal{O})/k(x)$ and $k(\mathcal{O})/k(y)$: $H_{(x,y)} = f+g$ with f fixed by G_x and g fixed by G_y.

Given a regular fraction $H(x,y)$, one could try to use the normal basis theorem (see [Lan02, chapter 6, § 13]) to test the existence of a Galois decoupling for $H(x,y)$. The normal basis theorem states that there exists a k_{inv}-basis of $k(\mathcal{O})$ of the form $(\sigma(\alpha))_{\sigma\in G}$ for some $\alpha\in k(\mathcal{O})$. The action of G_x and G_y on this basis is given by permutation matrices, and thus the linear constraints for the Galois decoupling of $H_{(x,y)}$ is equivalent to a system of linear equations. Unfortunately the computation of a normal basis requires *a priori* a complete knowledge of the Galois group G, whose computation is a difficult problem (see 3.1.4). Therefore, we present in the remaining of the section a construction of a Galois decoupling test which relies entirely on the orbit and its Galoisian structure.

4.2 The decoupling of (x,y) in the orbit

Definition 4.2.1. Let α be a 0-chain of the orbit. We say that α *cancels decoupled fractions* if $H_\alpha = 0$ for any regular fraction $H(x,y)$ of $\mathbb{C}(x,t)+\mathbb{C}(y,t)$.



$$\alpha = (u_1, v_2) - (u_1, v_1) + (u_2, v_3) - (u_2, v_2) + (u_3, v_1) - (u_3, v_3)$$

$$F(x)_\alpha = F(u_1) - F(u_1) + F(u_2) - F(u_2) + F(u_3) - F(u_3) = 0$$

$$G(y)_\alpha = G(v_2) - G(v_1) + G(v_3) - G(v_2) + G(v_1) - G(v_3) = 0$$

Figure 4.2.1 – The 0-chain induced by a bicolored cycle cancels decoupled fractions

We recall that a *path* in the graph of the orbit is a sequence of vertices $(a_1, a_2, \ldots, a_{n+1})$ such that $a_i \sim a_{i+1}$ for all $0 \leq i \leq n$. The length of $(a_1, a_2, \ldots, a_{n+1})$ is the number of adjacencies (that is n). A path is called a *cycle** if $a_{n+1} = a_1$. A cycle is called *simple* if only its first and last vertices are equal.

Example 4.2.2. A *bicolored cycle* is a cycle $(a_1, a_2, \ldots, a_{2n+1})$ of even length such that for all i, $a_{2i} \sim^x a_{2i-1}$ and $a_{2i+1} \sim^y a_{2i}$. One associates to $(a_1, a_2, \ldots, a_{2n+1})$ the 0-chain

$$\alpha = \sum_{i=1}^{2n} (-1)^i a_i = \sum_{i=1}^{n} (a_{2i} - a_{2i-1}) = \sum_{i=1}^{n} (a_{2i} - a_{2i+1}).$$

Taking $F(x,t)$ a fraction in $\mathbb{C}(x,t)$, one observes that for all i one has $F_{a_{2i}} - F_{a_{2i-1}} = 0$, as vertices a_{2i} and a_{2i-1} share their first coordinate. Symmetrically, taking $G(y,t)$ a fraction in $\mathbb{C}(y,t)$, one has $G_{a_{2i+1}} - G_{a_{2i}} = 0$. Therefore, $F_\alpha = G_\alpha = 0$. Hence, the *0-chains induced by bicolored cycles* cancel decoupled fractions. Figure 4.2.1 illustrates this observation. ∎

Example 4.2.2 is fundamental for picturing the 0-chains that cancel decoupled fractions because of the following stronger result:

Proposition 4.2.3. *A 0-chain cancels decoupled fractions if and only if it can be decomposed as a $\mathbb{C}$-linear*† *combination of 0-chains induced by bicolored cycles.*

There exists an elementary graph theoretic proof of this fact. However, we choose to postpone the proof of Proposition 4.2.3 after the proof of Theorem 4.4.10, which is an algebraic reformulation of the condition for a 0-chain to cancel decoupled fractions.

Example 4.2.4. A straightforward application of this observation, is the following obstruction for the existence of a Galois decoupling of xy. Consider an orbit whose graph contains a bicolored square (bicolored cycle of length 4), with associated 0-chain

*. The terminology "loop" of [Gib10, Definition 1.8] is unorthodox, so we write "cycle" instead.
†. Note that if the 0-chain is with integer coefficients, one can choose the combination with integer coefficients as well.

$\alpha = (u_1, v_1) - (u_1, v_2) + (u_2, v_2) - (u_2, v_1)$ (thus with $u_1 \neq u_2$ and $v_1 \neq v_2$). The evaluation of xy on this 0-chain factors as $(xy)_\alpha = (u_1 - u_2)(v_1 - v_2)$, which is always nonzero.

Therefore, if the orbit of a model $\mathcal{S}$ contains a bicolored square, then xy never admits a Galois decoupling and thus a decoupling in the sense of the t-equivalence. Thus, we can conclude that for models with orbit $\widetilde{\mathcal{O}_{12}}$ (see Figure 3.1.1) or Hadamard (see Section 4.6.2), or the "Fan model" (see Appendix 4.6.3), the fraction xy never admits a decoupling. ∎

For now, we only saw that the canceling of a regular fraction on 0-chains that cancel decoupled fraction gives a necessary condition for the Galois decoupling of this fraction. We prove in this section that this condition is in fact sufficient and that one only needs to consider the evaluation on a single 0-chain.

For small steps walks with finite orbit, there is only one bicolored cycle and thus only one 0-chain α induced by the bicolored cycle. Theorem 4.11 in [BBR21] shows that a regular fraction admits a Galois decoupling if and only its evaluation on α is zero. More precisely, Bernardi, Bousquet-Mélou and Raschel proved an explicit identity in the algebra of the group of the walk. Rephrasing their equality in terms of 0-chains in the orbit, we introduce the notion of decoupling of the pair (x, y) in the orbit as follows:

Definition 4.2.5 (Decoupling of (x, y))**.** We say that (x, y) *admits a decoupling in the orbit* if there exist 0-chains $\widetilde{\gamma_x}$, $\widetilde{\gamma_y}$, α such that

— $(x, y) = \widetilde{\gamma_x} + \widetilde{\gamma_y} + \alpha$

— $\sigma_x \cdot \widetilde{\gamma_x} = \widetilde{\gamma_x}$ for all $\sigma_x \in G_x$

— $\sigma_y \cdot \widetilde{\gamma_y} = \widetilde{\gamma_y}$ for all $\sigma_y \in G_y$

— the 0-chain α cancels decoupled fractions

In that case, we call the identity $(x, y) = \widetilde{\gamma_x} + \widetilde{\gamma_y} + \alpha$ a *decoupling of* (x, y).

Note that if $(\widetilde{\gamma_x}, \widetilde{\gamma_y}, \alpha)$ is a decoupling of (x, y) then the 0-chain α is equal to $(x, y) - \widetilde{\gamma_x} - \widetilde{\gamma_y}$. Hence, when giving such a decoupling, we will often state explicitly only $\widetilde{\gamma_x}$ and $\widetilde{\gamma_y}$.

Example 4.2.6. For the orbit of the model $\mathcal{G}_\lambda$, a decoupling equation $(x, y) = \widetilde{\gamma_x} + \widetilde{\gamma_y} + \alpha$ is given

$$\text{by } \widetilde{\gamma_x} = \Bigg(\frac{1}{2}\left((x,y) + (x,\overline{xy})\right) - \frac{1}{8}\left((z,y) + (-\overline{xy^2z},y) + (xy^2z,\overline{xy}) + (-\overline{z},\overline{xy})\right)$$
$$+\frac{1}{8}\left((xy^2z,\overline{yz}) + (z,\overline{yz}) + (-\overline{xy^2z},-xyz) + (-\overline{z},-xyz)\right)\Bigg)$$
$$\text{and } \widetilde{\gamma_y} = \left(\frac{1}{4}\left((x,y) + (z,y) + (-\overline{xy^2z},y)\right) - \frac{1}{4}\left((x,\overline{xy}) + (z,\overline{yz}) + (-\overline{xy^2z},-xyz)\right)\right),$$

where $\overline{\alpha}$ denotes $\frac{1}{\alpha}$. This decoupling is constructed in Subsection 4.6.4 and the 0-chain α is represented in Figure 4.2.2. It is the sum of the two 0-chains α_1 and α_2 induced by

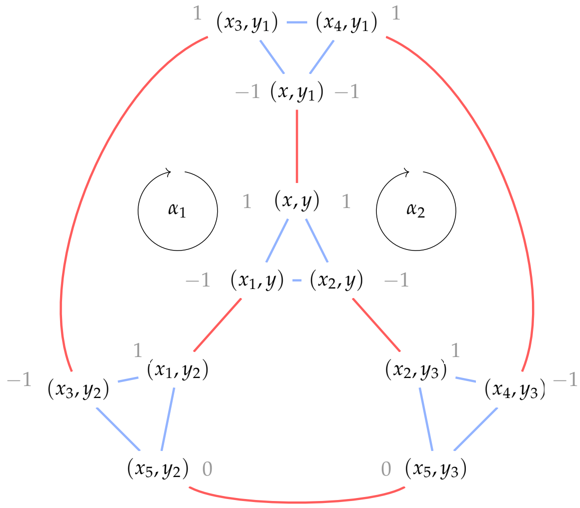


Figure 4.2.2 – A 0-chain of $\mathcal{O}_{12}$ characterizing decoupled fractions for the model $\mathcal{G}_\lambda$ where the weights are written in grey next to their corresponding vertex.

the two bicolored cycles where the weights of the 0-chains α_1 and α_2 are written in grey next to their corresponding vertex. ∎

The relation between the notion of decoupling of (x, y) in the orbit and the notion of Galois decoupling is detailed in the following proposition.

Proposition 4.2.7. *Assume that $(x, y) = \widetilde{\gamma_x} + \widetilde{\gamma_y} + \alpha$ is a decoupling of (x, y), and let $H(x, y)$ be a regular fraction. Then the following assertions are equivalent:*

(1) $H(x, y)$ admits a Galois decoupling

(2) $H_\alpha = 0$

(3) $H_{(x,y)} = H_{\widetilde{\gamma_x}} + H_{\widetilde{\gamma_y}}$ is a Galois decoupling of $H(x, y)$.

Proof. (3) $\Rightarrow$ (1) is obvious.

(1) $\Rightarrow$ (2): By definition of a Galois decoupling of (x, y), the 0-chain α cancels decoupled fractions.

(2) $\Rightarrow$ (3): Evaluating $H(x, y)$ on the decoupling of (x, y) yields the identity $H_{(x,y)} = H_{\widetilde{\gamma_x}} + H_{\widetilde{\gamma_y}}$. Moreover, since $\widetilde{\gamma_x}$ (resp. $\widetilde{\gamma_y}$) is fixed by G_x (resp. G_y), then Lemma 3.1.19 and the Galois correspondence in the extensions $k(\mathcal{O})/k(x)$ and $k(\mathcal{O})/k(y)$ ensure that $H_{\widetilde{\gamma_x}}$ and $H_{\widetilde{\gamma_y}}$ belong respectively to $k(x)$ and $k(y)$, hence $H_{\widetilde{\gamma_x}} + H_{\widetilde{\gamma_y}}$ is a Galois decoupling of $H(x, y)$. □

Therefore, if we solve the decoupling problem of (x, y) in the orbit, we also solve the Galois decoupling problem for regular fractions: an explicit decoupling of (x, y) will grant us with a simple test to check whether a regular fraction admits a Galois decoupling (some orbit sum is zero), and an effective way to construct the associated Galois decoupling based on orbit sum computations. We now state the main result of this section, whose proof will follow from Theorem 4.4.12.

Theorem 4.2.8 (Decoupling)**.** *If the orbit $\mathcal{O}$ is finite, then (x, y) always admits a decoupling in the orbit with rational coefficients.*

The remaining of this section is dedicated to the proof of Theorem 4.2.8 and to the effective construction of the decoupling of (x, y) in the orbit.

4.3 Pseudo-decoupling

We define here a more flexible notion of decoupling in the orbit called *pseudo-decoupling*, mainly used in the proof of the Theorem 4.2.8.

Definition 4.3.1 (Pseudo-decoupling)**.** Let γ_x and γ_y be two 0-chains. We call the pair (γ_x, γ_y) a *pseudo-decoupling* of (x, y) if for every regular fraction $H(x, y)$ that admits a Galois decoupling, the equation $H_{(x,y)} = H_{\gamma_x} + H_{\gamma_y}$ is a Galois decoupling of $H(x, y)$, that is, $H_{\gamma_x} \in k(x)$ and $H_{\gamma_y} \in k(y)$.

For instance, if $(x, y) = \widetilde{\gamma_x} + \widetilde{\gamma_y} + \alpha$ is a decoupling of (x, y), then the pair $(\widetilde{\gamma_x}, \widetilde{\gamma_y})$ is a pseudo-decoupling of (x, y) by Proposition 4.2.7.

Theorem 4.3.2 below shows how a pseudo-decoupling yields a decoupling. First let us give some notation. Let G' be a subgroup of G. We denote by $[G']$ the formal sum $\frac{1}{|G'|} \sum_{\sigma \in G'} \sigma$. From a Galois theoretic point of view, if G' is the Galois group of some subextension $k(\mathcal{O})/M$, then $[G']$ is the trace of the field extension $k(\mathcal{O})/M$.

Theorem 4.3.2. *If a pair (γ_x, γ_y) is a pseudo-decoupling of (x, y), then (x, y) admits a decoupling $(x, y) = \widetilde{\gamma_x} + \widetilde{\gamma_y} + \alpha$ where $\widetilde{\gamma_x} = [G_x] \cdot \gamma_x$ and $\widetilde{\gamma_y} = [G_y] \cdot \gamma_y$.*

Proof. By construction, the 0-chains $\widetilde{\gamma_x}$ and $\widetilde{\gamma_y}$ are fixed under the respective actions of G_x and G_y. Therefore, we only need to prove that α cancels decoupled fractions, for which purpose we rewrite it as the sum of three terms

$$\alpha = ((x, y) - \gamma_x - \gamma_y) + (\gamma_x - [G_x] \cdot \gamma_x) + (\gamma_y - [G_y] \cdot \gamma_y).$$

Let $H(x, y)$ be a regular fraction that admits a Galois decoupling. Then

— $H_{(x,y)} - H_{\gamma_x} - H_{\gamma_y} = 0$ by definition of the pseudo-decoupling (γ_x, γ_y).

— For σ_x in G_x, we compute $H_{\gamma_x - \sigma_x \cdot \gamma_x} = H_{\gamma_x} - \sigma_x(H_{\gamma_x})$ thanks to Lemma 3.1.19. As H_{γ_x} is in $k(x)$, it turns out that $H_{\gamma_x - \sigma_x \cdot \gamma_x}$ is zero.
Since $\frac{1}{|G_x|} \sum_{\sigma_x \in G_x} (\gamma_x - \sigma_x \cdot \gamma_x) = \gamma_x - [G_x] \cdot \gamma_x$, we obtain that $\gamma_x - [G_x] \cdot \gamma_x$ cancels the fraction $H(x, y)$.

The argument for $\gamma_y - [G_y] \cdot \gamma_y$ is similar. Thus $H_\alpha = 0$, which concludes the proof. □

We finish this subsection with two important lemmas.

Lemma 4.3.3. *If the pair (γ_x, γ_y) is a pseudo-decoupling of (x, y), and α and α' are 0-chains that cancel decoupled fractions, then $(\gamma_x + \alpha, \gamma_y + \alpha')$ is also a pseudo-decoupling of (x, y).*

Proof. Let $H(x,y)$ be a regular fraction that admits a Galois decoupling. By definition of α and α', we have $H_\alpha = H_{\alpha'} = 0$, which by linearity proves that $H_{\gamma_x+\alpha} = H_{\gamma_x}$ and $H_{\gamma_y+\alpha'} = H_{\gamma_y}$.

Since (γ_x, γ_y) is a pseudo-decoupling of (x,y), the equation

$$H_{(x,y)} = H_{\gamma_x} + H_{\gamma_y} = H_{\gamma_x+\alpha} + H_{\gamma_x+\alpha'}$$

is a Galois decoupling of $H(x,y)$ proving that $(\gamma_x + \alpha, \gamma_y + \alpha')$ is also a pseudo-decoupling of (x,y). □

Lemma 4.3.4. *If two* 0*-chains* γ_x *and* γ_y *satisfy the following conditions*

— $(x,y) = \gamma_x + \gamma_y$

— *for all* $\sigma_x \in G_x$*, the* 0*-chain* $\sigma_x \cdot \gamma_x - \gamma_x$ *cancels decoupled fractions*

— *for all* $\sigma_y \in G_y$*, the* 0*-chain* $\sigma_y \cdot \gamma_y - \gamma_y$ *cancels decoupled fractions*

then (γ_x, γ_y) *is a pseudo-decoupling of* (x,y)*.*

Proof. Let $H(x,y)$ be a regular fraction which admits a Galois decoupling. As $H_{(x,y)} = H_{\gamma_x} + H_{\gamma_y}$ from the first point, one only needs to show that H_{γ_x} is in $k(x)$ and that H_{γ_y} is in $k(y)$. Let σ_x be in G_x, then $\sigma_x(H_{\gamma_x}) = H_{\sigma_x\cdot\gamma_x} = H_{(\sigma_x\cdot\gamma_x-\gamma_x)+\gamma_x} = H_{\gamma_x}$ because $(\sigma_x \cdot \gamma_x - \gamma_x)$ cancels decoupled fractions. Therefore, the Galois correspondence proves that H_{γ_x} is in $k(x)$. The same argument proves that H_{γ_y} is in $k(y)$. □

4.4 Graph homology and construction of the decoupling

Our construction of a decoupling relies on the graph structure of the orbit $\mathcal{O}$, and in particular on the formalism of graph homology.

4.4.1 Basic graph homology

We recall here the basic definitions of graph homology and the properties that will be used in the construction of the decoupling (see [Gib10] for a comprehensive introduction to graph homology).

Definition 4.4.1. A *graph* (undirected) is a pair $\Gamma = (V, E)$ where V is the set of vertices and $E \subset \{\{a, a'\} : a, a' \in V, a \neq a'\}$ is the set of edges. A *subgraph* of Γ is a graph $\Gamma' = (V', E')$ such that $V' \subset V$ and $E' \subset E$.

An *oriented graph* is a pair $\Gamma = (V, E^+)$ where V is the set of vertices and $E^+ \subset \{(a, a') : a, a' \in V, a \neq a'\}$ the set of *arcs* (oriented edges) such that if $(a, a') \in E^+$ then $(a', a) \notin E^+$. An *orientation* of a graph $\Gamma = (V, E)$ is an oriented graph $\Gamma' = (V, E^+)$ such that the map $E^+ \to E$ which maps (a, a') to $\{a, a'\}$ is a bijection.

Note that every graph can be given an orientation by freely choosing an origin for each edge. Conversely, given an oriented graph $\Gamma = (V, E^+)$, one can consider the associated undirected graph (V, E) where $E = \{\{a, a'\}$ such that $(a, a') \in E^+$ or $(a', a) \in E^+\}$. In what follows, the notions of graph homomorphism, path, connected components concern the structure of undirected graph.

Example 4.4.2. The graphs considered here are the graph induced by the orbit $(\mathcal{O}, \sim)$ still denoted $\mathcal{O}$, and the two subgraphs of the orbit restricted to each individual type of adjacency, which are $\mathcal{O}^x = (\mathcal{O}, \sim^x)$ and $\mathcal{O}^y = (\mathcal{O}, \sim^y)$. ∎

We now introduce the chain complex attached to an oriented graph.

Definition 4.4.3. Let $\Gamma = (V, E^+)$ be an oriented graph and K a field. Such a graph induces a chain complex $C_*(\Gamma)$ defined as follows. The space $C_0(\Gamma)$ of 0-chains of Γ is the free K-vector space spanned by the vertices of V. Similarly, the space $C_1(\Gamma)$ of 1-chains of Γ is the free K-vector space spanned by the arcs of E^+. The *boundary homomorphism* is then the K-linear map ∂ defined by

$$\begin{array}{rccc} \partial\colon & C_1(\Gamma) & \longrightarrow & C_0(\Gamma) \\ & (a, a') \in E^+ & \longmapsto & a' - a \end{array}$$

As the reader notices, the chain complex has only been defined for an oriented graph. Nonetheless, if (V, E_1^+) and (V, E_2^+) are two orientations of a graph Γ, it is easy to see that the associated chain complexes are isomorphic [Gib10, 1.21 (3)]. When the context is clear, we shall abuse notation and define a chain complex $C_*(\Gamma)$ of a graph Γ as the chain complex of the oriented graph (V, E^+) where E^+ is an arbitrary orientation of Γ.

We make the following convention. Let a and a' be two adjacent vertices of Γ. Given an orientation E^+ of Γ, we abuse notation and denote by (a, a') the 1-chain

$$(a, a') = \begin{cases} (a, a') & \text{if } (a, a') \text{ is in } E^+ \\ -(a', a) & \text{otherwise} \end{cases}.$$

This notation is extremely convenient, because for two adjacent vertices of Γ, the boundary homomorphism always satisfies $\partial((a, a')) = a' - a$ and $(a, a') = -(a', a)$.

Definition 4.4.4. Let $\Gamma = (V, E^+)$ be an oriented graph. A 1-chain c which satisfies $\partial(c) = 0$ is called a 1-*cycle*.

Example 4.4.5 (1-chain induced by a path)**.** Let $\Gamma = (V, E)$ be a graph and let $(a_1, a_2, \ldots, a_{n+1})$ be a path in Γ, that is, a sequence of vertices such that a_i is adjacent to a_{i+1} for $i = 1, \ldots, n$.

Given an arbitrary orientation E^+ of Γ, we define the 1-chain $p = \sum_{i=1}^{n} (a_i, a_{i+1})$, and we call it the 1-*chain induced by the path* $(a_1, a_2, \ldots, a_{n+1})$.

By telescoping, $\partial(p) = a_{n+1} - a_1$, therefore if the path is a cycle of Γ then p is a 1-cycle, hence the name. Every 1-cycle is a linear combination of 1-cycles induced by the simple cycles of the graph [Gib10, Theorem 1.20]). ∎

We recall that a graph is called *connected* if any two vertices are joined by a path. The reader should note that the notion of path does not take into account a potential orientation of the edges.

Every finite graph is the disjoint union of finitely many connected components which are maximal connected subgraphs. Any orientation of a graph induces by restriction an orientation on its subgraphs and thus on its connected components.

With this convention, it turns out that the chain complex of a finite oriented graph is isomorphic to the direct sum of the chain complexes of its connected components. Hence, it is harmless to extend Theorem 1.23 in [Gib10] to the case of a non-connected graph.

Proposition 4.4.6. *Let* $\Gamma = (V, E)$ *be a graph, and let* $(\Gamma_i = (V_i, E_i))_{i=1,\dots,r}$ *be its connected components. Define the* augmentation map

$$\begin{aligned} \varepsilon \colon C_0(\Gamma) &\longrightarrow K^r \\ \sum_{a \in V} \lambda_a \, a &\longmapsto \left(\sum_{a \in V_i} \lambda_a \right)_{i=1,\dots,r} \end{aligned} .$$

Then, $\operatorname{Ker} \varepsilon = \operatorname{Im} \partial$*.*

Let $\Gamma = (V, E)$ be a graph and let σ be a graph endomorphism of Γ. Fixing an orientation E^+ on Γ, we let σ act on the space of 0 and 1-chains by K-linearity via:

$$\sigma \cdot a = \sigma(a) \text{ for any } a \text{ in } V \text{ and } \sigma \cdot (a, a') = (\sigma(a), \sigma(a')) \text{ for any } (a, a') \text{ in } E^+.$$

The reader should note that the action on the space of 1-chains uses the convention on the arc notation introduced at the beginning of this subsection. It is easily seen that the action of a graph endomorphism of Γ is compatible with the boundary homomorphism of the chain complex $C_*(\Gamma)$.

Proposition 4.4.7. *Let* $\Gamma = (V, E)$ *be a graph and* σ *be a graph endomorphism of* Γ*. Then* σ *induces a* chain map *on* $C_*(\Gamma)$*, which means that the following diagram of* K*-linear maps is commutative.*

$$\begin{array}{ccc} C_1(\Gamma) & \xrightarrow{\partial} & C_0(\Gamma) \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma} \\ C_1(\Gamma) & \xrightarrow{\partial} & C_0(\Gamma) \end{array}$$

4.4.2 The chain complex of the orbit

We now apply the homological formalism to the graphs associated with the orbit $\mathcal{O}$ with base field $\mathbb{C}$ (see Example 4.4.2). We fix once for all an orientation on $\mathcal{O}$ which induces an orientation on the subgraphs $\mathcal{O}^x$ and $\mathcal{O}^y$. Quoting [Gib10, Remark 1.21], “the choice of this orientation is just a technical device introduced to enable the computation of the boundary homomorphisms”.

We denote by ∂ (resp. ∂^x, ∂^y) the boundary homomorphism on the connected graph $\mathcal{O}$ (resp. the non-connected graphs $\mathcal{O}^x$, $\mathcal{O}^y$). Moreover, we denote by ε (resp. ε^x, ε^y) the augmentation map defined in Proposition 4.4.6 for $\mathcal{O}$ (resp. $\mathcal{O}^x$, $\mathcal{O}^y$).

Lemma 4.4.8. *The $\mathbb{C}$-vector space $C_1(\mathcal{O})$ is equal to $C_1(\mathcal{O}^x) \oplus C_1(\mathcal{O}^y)$ and the boundary homomorphism ∂ coincides with $\partial^x + \partial^y$ where one has extended ∂^x (resp. ∂^y) by zero on $C_1(\mathcal{O}^y)$ (resp. $C_1(\mathcal{O}^x)$).*

Proof. Every edge $\{a, a'\}$ of $\mathcal{O}$ is either an x-adjacency or a y-adjacency, and not both. Therefore, the set of arcs of an orientation of $\mathcal{O}$ is the disjoint union of the arcs of the orientations of $\mathcal{O}^x$ and $\mathcal{O}^y$, which thus induces a direct sum decomposition on the free vector space $C_1(\mathcal{O})$. The decomposition of the homomorphism ∂ follows directly. □

The action of the Galois group G on the vertices of $\mathcal{O}$ preserves the adjacency types of the edges (see Lemma 3.1.5). Therefore G acts by graph automorphisms on $\mathcal{O}^x$ and $\mathcal{O}^y$. Thus, Proposition 4.4.7 allows us to define the action of G on the chains of $\mathcal{O}^x$ and $\mathcal{O}^y$ in a compatible way with the decomposition of Lemma 4.4.8.

Proposition 4.4.9. *Let σ be in G. Then σ induces automorphisms of the chain complexes $C_*(\mathcal{O})$, $C_*(\mathcal{O}^x)$ and $C_*(\mathcal{O}^y)$ such that $\sigma \circ \partial^x = \partial^x \circ \sigma$ and $\sigma \circ \partial^y = \partial^y \circ \sigma$.*

The boundary homomorphisms ∂^x and ∂^y allow us to rewrite the 0-chains induced by bicolored cycles as boundaries. If α is the 0-chain associated to a bicolored cycle as in Example 4.2.2, then it is easily seen that $\alpha = \partial^x(p) = \partial^y(-p)$ with p the 1-chain as in Example 4.4.5. The homology formalism generalizes the above description to any 0-chain that cancels decoupled fractions.

Theorem 4.4.10. *Let α be a 0-chain. Then the following statements are equivalent:*

(1) α cancels decoupled fractions.

(2) $\varepsilon^x(\alpha) = 0$ and $\varepsilon^y(\alpha) = 0$.

(3) There exists a 1-cycle c of $\mathcal{O}$ such that $\alpha = \partial^x(c)$.

(3') There exists a 1-cycle c of $\mathcal{O}$ such that $\alpha = \partial^y(c)$.

Proof. (1) $\Rightarrow$ (2): Let α be a 0-chain that cancels decoupled fractions. The connected components of the graph $\mathcal{O}^x$ are of the form $\mathcal{O}^x_u = \{(u', v') \in \mathcal{O} : u' = u\}$ for the distinct left coordinates u of $\mathcal{O}$. We denote by U the set of distinct left coordinates. Therefore, we decompose $\alpha = \sum_{u \in U} \alpha_u$ where $\alpha_u = \sum_{v'} \lambda^u_{v'} (u, v')$ is a 0-chain with vertices in $\mathcal{O}^x_u$. Now, we consider the family of monomials $(x^i)_i$ which are obviously decoupled. Since α cancels decoupled fractions, the following holds for all i:

$$0 = (x^i)_\alpha = \sum_u (x^i)_{\alpha_u} = \sum_u \sum_{v' \,/\, (u,v') \in \mathcal{O}^x_u} \lambda^u_{v'} (x^i)_{(u,v')} = \sum_u \left(\sum_{v' \,/\, (u,v') \in \mathcal{O}^x_u} \lambda^u_{v'} \right) u^i = \sum_u \varepsilon^x(\alpha)_u u^i.$$

The vector $(\varepsilon^x(\alpha)_u)_{u \in U}$ lies therefore in the kernel of the Vandermonde matrix $(u^i)_{i<|U|, u \in U}$. Since the elements of U are distinct, this matrix is invertible and the $\varepsilon^x(\alpha)_u$ are all equal to 0. Thus, $\varepsilon^x(\alpha) = 0$. The same argument yields $\varepsilon^y(\alpha) = 0$.

(2) $\Rightarrow$ (3) and (3'): Assume that $\varepsilon^x(\alpha) = 0$ and $\varepsilon^y(\alpha) = 0$. By Proposition 4.4.6, there exist c_x in $C_1(\mathcal{O}^x)$ and c_y in $C_1(\mathcal{O}^y)$ such that $\partial^x(c_x) = \alpha$ and $\partial^y(c_y) = \alpha$. Moreover,

$$\partial(c_x - c_y) = \partial(c_x) - \partial(c_y) = \partial^x(c_x) - \partial^y(c_y) = \alpha - \alpha = 0.$$

Therefore, $c = c_x - c_y$ is a 1-cycle of $\mathcal{O}$ which satisfies $\partial^x(c) = \alpha$ and $\partial^y(-c) = \alpha$.

(3) $\Leftrightarrow$ (3′): Let c be a 1-cycle of $\mathcal{O}$, then $\partial^x(c) = \partial(c) - \partial^y(c) = \partial^y(-c)$. This proves the equivalence.

(3) $\Rightarrow$ (1): Assume that $\alpha = \partial^x(c) = \partial^y(-c)$ for c a cycle of $\mathcal{O}$. Now, let $e = ((u,v),(u,v'))$ be an arc of $\mathcal{O}^x$ and take $F(x) \in \mathbb{C}(x,t)$. Then

$$F_{\partial^x(e)} = F(u, 1/S(x,y)) - F(u, 1/S(x,y)) = 0.$$

Therefore, by $\mathbb{C}$-linearity, this implies that $F_\alpha = F_{\partial^x(c)} = 0$. Symmetrically, if $G(y) \in \mathbb{C}(y,t)$, then we deduce that $G_\alpha = G_{\partial^y(-c)} = 0$, which concludes the proof. $\square$

We now apply this pleasant characterization to prove our earlier claim that 0-chains that cancel decoupled fractions are induced by $\mathbb{C}$-linear combinations of 1-cycles induced by bicolored cycles.

Proof of Proposition 4.2.3. Let α be a 0-chain which cancels decoupled fractions, then by (3) of Theorem 4.4.10, we can write it $\alpha = \partial^x(c) = -\partial^y(c)$ with c a 1-cycle of $\mathcal{O}$. Since the 1-cycles induced by the simple cycles of $\mathcal{O}$ generate the 1-cycles of $\mathcal{O}$ (see [Gib10, Theorem 1.20]), we can assume without loss of generality that c is induced by a simple cycle $p = (a_1, a_2, \ldots, a_n)$ of $\mathcal{O}$.

Moreover, if consecutive arcs $e_i, \ldots, e_{i+k-1} = (a_i, a_{i+1}), (a_{i+1}, a_{i+2}), \ldots, (a_{i+k-1}, a_{i+k})$ of p are of the same adjacency type (say x), then since the monochromatic components of $\mathcal{O}$ are cliques, (a_i, a_{i+k}) is an arc of $\mathcal{O}$. Therefore,

$$\partial^x(e_i + \cdots + e_{i+k-1}) = \partial^x(e_i + \cdots + e_{i+k-1} + (a_{i+k}, a_i)) + \partial^x((a_i, a_{i+k})),$$

the first term being zero because it is the boundary of a monochromatic cycle.

The exact same reasoning can be done for consecutive y-adjacencies. Thus, replacing consecutive arcs of the same adjacency type by one single arc of the same adjacency type, we can assume without loss of generality that c is the 1-chain induced by a simple bicolored cycle. This proves that α is the 0-chain induced by a bicolored cycle, finishing the proof. $\square$

4.4.3 Construction of the decoupling

We now use the results of the previous subsections to construct a pseudo-decoupling of (x,y) on a finite orbit $\mathcal{O}$. For $p = (p_a)_{a\in\mathcal{O}}$ a family of 1-chains, we consider the 0-chains

$$\gamma_x(p) = -\frac{1}{|\mathcal{O}|}\sum_a \partial^y(p_a) \text{ and } \gamma_y(p) = -\frac{1}{|\mathcal{O}|}\sum_a \partial^x(p_a)$$

where all sums run over $\mathcal{O}$. The $\mathbb{C}$-linearity of the boundary homomorphisms implies that γ_x and γ_y are $\mathbb{C}$-linear.

We recall that ω is the 0-chain $\frac{1}{|\mathcal{O}|}\sum_{a\in\mathcal{O}} a$ defined in Lemma 3.2.5.

Lemma 4.4.11. *Let $(c_a)_{a\in\mathcal{O}}$ be a family of 1-chains. Then the following assertions hold.*

1. *If $\partial(c_a) = 0$ for all a, then the two 0-chains $\gamma_x(c)$ and $\gamma_y(c)$ cancel decoupled fractions.*
2. *If $c_a \in C_1(\mathcal{O}^x)$ for all a, then $\gamma_x(c) = 0$. Likewise, if $c_a \in C_1(\mathcal{O}^y)$ for all a, then $\gamma_y(c) = 0$.*

Proof. We prove the first point. Assume that $\partial(c_a) = 0$ for all a, then by Theorem 4.4.10 the 0-chains $\partial^x(c_a)$ and $\partial^y(c_a)$ cancel decoupled fractions for all a. Hence, by linearity, so do $\gamma_x(c)$ and $\gamma_y(c)$. We prove the second point. Assume that $c_a \in C_1(\mathcal{O}^x)$ for all a, then $\partial^y(c_a) = 0$ for all a. Thus, $\gamma_x(c) = 0$ by linearity. The case $c_a \in C_1(\mathcal{O}^y)$ is symmetric. ☐

Theorem 4.4.12 (Decoupling theorem). *Let $p^x = (p_a^x)_{a\in\mathcal{O}}$ and $p^y = (p_a^y)_{a\in\mathcal{O}}$ be two families of 1-chains, that are such that, for all $a \in \mathcal{O}$, one has*

$$\varepsilon^x(\partial(p_a^x) + (x,y) - a) = 0 \text{ and } \varepsilon^y(\partial(p_a^y) + (x,y) - a) = 0. \tag{4.4.1}$$

Then, the pair $(\omega + \gamma_x(p^x), \gamma_y(p^y))$ is a pseudo-decoupling of (x,y).

Proof. Let us first consider a pair (p^x, p^y) of families of 1-chains such that $p^x = p^y$ and $\partial(p_a^x) + (x,y) - a = 0$ for all a in $\mathcal{O}$. In order to prove that $(\omega + \gamma_x(p^x), \gamma_y(p^y))$ is a pseudo-decoupling, we will show that this pair satisfies the three conditions of Lemma 4.3.4.

The first condition comes down to showing that $(x,y) = \omega + \gamma_x(p^x) + \gamma_y(p^y)$. By construction, we have that

$$a - (x,y) = \partial(p_a^x) = \partial^y(p_a^x) + \partial^x(p_a^x),$$

for all a in $\mathcal{O}$. Summing this identity over the orbit yields

$$\sum_{a\in\mathcal{O}} a - |\mathcal{O}| \cdot (x,y) = \sum_{a\in\mathcal{O}} \partial^y(p_a^x) + \sum_{a\in\mathcal{O}} \partial^x(p_a^x) = \sum_{a\in\mathcal{O}} \partial^y(p_a^x) + \sum_{a\in\mathcal{O}} \partial^x(p_a^y),$$

which can be rewritten as $(x,y) = (\omega + \gamma_x(p^x)) + \gamma_y(p^y)$.

For the second condition, we need to prove that $\sigma_x \cdot (\omega + \gamma_x(p^x)) - (\omega + \gamma_x(p^x))$ cancels decoupled fractions for every σ_x in G_x. The compatibility of G with the boundary homomorphisms (Proposition 4.4.9) yields

$$\begin{aligned}\sigma_x \cdot \gamma_x(p^x) &= -\frac{1}{|\mathcal{O}|}\sum_{a\in\mathcal{O}} \partial^y(\sigma_x \cdot p_a^x) \\ &= -\frac{1}{|\mathcal{O}|}\sum_{a\in\mathcal{O}} \partial^y(p^x_{\sigma_x\cdot a}) - \frac{1}{|\mathcal{O}|}\sum_{a\in\mathcal{O}} \partial^y(\sigma_x \cdot p_a^x - p^x_{\sigma_x\cdot a}).\end{aligned}$$

The homomorphism σ_x induces a bijection on the vertices of $\mathcal{O}$, so the first sum on the right hand-side is equal to $\gamma_x(p^x)$. Hence, we find

$$\sigma_x \cdot \gamma_x(p^x) - \gamma_x(p^x) = -\frac{1}{|\mathcal{O}|}\sum_{a\in\mathcal{O}} \partial^y(\sigma_x \cdot p_a^x - p^x_{\sigma_x\cdot a}).$$

Moreover, since σ_x fixes $x \in k(\mathcal{O})$, we have $\sigma_x \cdot (x,y) = (x,v)$ for some v. Thus, the arc $c =: ((x,y),(x,v)) \in C_1(\mathcal{O}^x)$ satisfies $\partial(c) = \sigma_x \cdot (x,y) - (x,y)$ and $\partial^y(c) = 0$. Hence, we may rewrite the above equation into

$$\sigma_x \cdot \gamma_x(p^x) - \gamma_x(p^x) = -\frac{1}{|\mathcal{O}|} \sum_{a \in \mathcal{O}} \partial^y(\sigma_x \cdot p_a^x - p_{\sigma_x \cdot a}^x + c). \tag{4.4.2}$$

By assumption on p^x, we have $\partial(p_a^x) = a - (x,y)$ for all a in $\mathcal{O}$. The compatibility of G_x with the boundary homomorphism yields for all $a \in \mathcal{O}$ that

$$\begin{aligned} \partial(\sigma_x \cdot p_a^x - p_{\sigma_x \cdot a}^x + c) &= \sigma_x(\partial(p_a^x)) - \partial(p_{\sigma_x \cdot a}^x) + \partial(c) \\ &= \sigma_x \cdot (a - (x,y)) - (\sigma_x \cdot a - (x,y)) + (\sigma_x \cdot (x,y) - (x,y)) = 0. \end{aligned}$$

Therefore, the 1-chain $\sigma_x \cdot q_a - q_{\sigma_x \cdot a} + c$ is a 1-cycle for all a, so, by Theorem 4.4.10, the 0-chain $\partial^y(\sigma_x \cdot p_a^x - p_{\sigma_x \cdot a}^x + c)$ cancels decoupled fractions. By linearity, we conclude from (4.4.2) that $\sigma_x \cdot \gamma_x(p^x) - \gamma_x(p^x)$ cancels decoupled fractions. Finally, as ω is fixed by σ_x, we deduce that $\sigma_x \cdot (\omega + \gamma_x(p^x)) - (\omega + \gamma_x(p^x)) = \sigma_x \cdot \gamma_x(p^x) - \gamma_x(p^x)$ cancels decoupled fractions. The proof that $(\omega + p^x, p^y)$ satisfies the third condition of Lemma 4.3.4 is completely analogous to the proof for the second condition.

Let us now prove the general case by showing that one can always reduce to the situation above. Let $(p_a^x)_{a \in \mathcal{O}}$ and $(p_a^y)_{a \in \mathcal{O}}$ be two families of 1-chains that satisfy (4.4.1).

From Proposition 4.4.6 applied to the graphs $\Gamma = \mathcal{O}^x$ and $\Gamma = \mathcal{O}^y$, we see that (4.4.1) is equivalent to the existence of two families of 1-chains $\delta^x = (\delta_a^x)_{a \in \mathcal{O}}$ and $\delta^y = (\delta_a^y)_{a \in \mathcal{O}}$ with δ_a^x in $C_1(\mathcal{O}^x)$ and δ_a^y in $C_1(\mathcal{O}^y)$ such that for all $a \in \mathcal{O}$ one has

$$\partial(p_a^x - \delta_a^x) = a - (x,y) \text{ and } \partial(p_a^y - \delta_a^y) = a - (x,y). \tag{4.4.3}$$

Define a family of 1-chains $q^x = (q_a^x)_{a \in \mathcal{O}}$ by $q_a^x =: p_a^x - \delta_a^x$. By construction, $\partial(q_a^x) = a - (x,y)$. The first part of the proof shows that the pair $(\omega + \gamma_x(q^x), \gamma_y(q^x))$ is a pseudo-decoupling. Lemma 4.4.11 (2) yields $\gamma_x(p^x) = \gamma_x(q^x)$ and $\gamma_y(p^y) = \gamma_y(p^y - \delta^y)$. Moreover, (4.4.3) implies that

$$\partial((p_a^y - \delta_a^y) - q_a^x) = \partial((p_a^y - \delta_a^y) - (p_a^x - \delta_a^x)) = a - (x,y) - (a - (x,y)) = 0,$$

for all $a \in \mathcal{O}$. By Lemma 4.4.11 (1), we obtain that $\gamma_y(p^y) - \gamma_y(q^x) = \gamma_y((p^y - \delta^y) - q^x)$ cancels decoupled fractions.

Thus, the pairs of 0-chains $(\omega + \gamma_x(p^x), \gamma_y(p^y))$ and $(\omega + \gamma_x(q^x), \gamma_y(q^x))$ differ by 0-chains that cancel decoupled fractions. Lemma 4.3.3 concludes the proof. □

We can now prove the existence of a decoupling of (x,y) for any finite orbit.

Proof of Theorem 4.2.8. The graph $\mathcal{O}$ is connected. Hence, for every $a \in \mathcal{O}$, there exists a path from (x,y) to a. Denoting by $p_a^x = p_a^y$ the associated 1-chain, we have $\partial(p_a^x) = a - (x,y)$ (see Example 4.4.5).

Therefore, the families $(p_a^x)_{a \in \mathcal{O}}$ and $(p_a^y)_{a \in \mathcal{O}}$ satisfy the assumptions of Theorem 4.4.12 leading to the existence of a pseudo-decoupling. Theorem 4.3.2 establishes the existence of a decoupling obtained from a pseudo-decoupling concluding the proof of

Theorem 4.2.8: if the orbit is finite, the pair (x, y) always admits a decoupling in the orbit. □

The reader may have noticed that we used in the proof of Theorem 4.2.8 two families of 1-chains p^x, p^y that satisfy a stronger assumption than (4.4.1). The idea beyond the assumption (4.4.1) is that it offers more flexibility in the choice of the families of 1-chains p^x, p^y. Such flexibility is required in Theorem 4.5.8. Indeed, there is a precise choice of the families of 1-chains that produces a pseudo-decoupling on which the evaluation of regular fractions is more efficient.

In [BEFHR25, Definition 6.1], the authors introduce the notion of a multiplicative decoupling of a regular fraction. In our context, we say that a regular fraction $H(x, y)$ has a multiplicative Galois decoupling if and only if there exists a positive integer m such that

$$H(x, y)^m = F(x)G(y) + \widetilde{K}(x, y)P(x, y),$$

for some rational fractions $F(x)$, $G(y)$ and a regular fraction $P(x, y)$.

Theorem 4.2.8 yields a decoupling of (x, y) with 0-chains $\widetilde{\gamma_x}, \widetilde{\gamma_y}$ and α having rational coefficients. Let d be the common denominator of the rational coefficients of $\widetilde{\gamma_x}, \widetilde{\gamma_y}$ and α which is easily seen to divide the size of the orbit in the proof of Theorem 4.4.12 when the input 1-chains in p^x and p^y all have integer coefficients. Then, the 0-chains $d\widetilde{\gamma_x}, d\widetilde{\gamma_y}, d\alpha$ have integer coefficients.

For such chains, one can define a multiplicative evaluation. For a 0-chain $c = \sum_{u,v} c_{u,v}(u, v)$ with integer coefficients, define

$$H_c^{\text{mul}} = \prod_{u,v} H(u, v, 1/S(x, y))^{c_{u,v}}.$$

As a direct corollary of the existence of a decoupling in the orbit, the following lemma gives an explicit procedure to test and construct, when it exists, the multiplicative Galois decoupling of a regular fraction $H(x, y)$.

Lemma 4.4.13. *The following statements are equivalent:*

— *$H(x, y)$ has a multiplicative Galois decoupling.*

— *There exists a positive integer m such that $\left(H_{d\alpha}^{mul}\right)^m = 1$.*

Proof. By definition, and from Proposition 3.1.18, the regular fraction $H(x, y)$ admits a multiplicative Galois decoupling if and only if there exist a positive integer m, $f(x) \in k(x)$ and $g(y) \in k(y)$ such that $H^m_{(x,y)} = f(x)g(y)$.

Let us assume that $H(x, y)$ admits a multiplicative Galois decoupling and let m be a positive integer such that $H(x, y)^m = F(x)G(y) + K(x, y)P(x, y)$ for some fractions $F(x)$, $G(x)$ in $\mathbb{C}(x, t)$ and a regular fraction $P(x, y) \in \mathbb{C}(x, y, t)$.

By multiplicative evaluation of the previous identity on $d\alpha$, we find that

$$\left(H_{d\alpha}^{\text{mul}}\right)^m = \left(F(x)_{d\alpha}^{\text{mul}}\right)^m \left(G(y)_{d\alpha}^{\text{mul}}\right)^m.$$

It is clear that $d\alpha$ is a 0-chain with integer coefficients that cancels decoupled fractions. By Proposition 4.2.3, the chain $d\alpha$ is a $\mathbb{Z}$-linear combination of 0-chains induced by bicolored cycles. One proves easily by a multiplicative analogue of Example 4.2.2 that if β is a 0-chain induced by a bicolored cycle then $F(x)^{\text{mul}}_{\beta} = G(y)^{\text{mul}}_{\beta} = 1$ which concludes the proof of the first implication.

Conversely, if there exists a positive integer m such that $\left(H^{\text{mul}}_{d\alpha}\right)^m = 1$, the decoupling $d \cdot (x,y) = d\widetilde{\gamma_x} + d\widetilde{\gamma_y} + d\alpha$ yields by multiplicative evaluation

$$\left(H^{\text{mul}}_{d(x,y)}\right)^m = H_{(x,y)}{}^{dm} = \left(H^{\text{mul}}_{d\widetilde{\gamma_x}}\right)^m \left(H^{\text{mul}}_{d\widetilde{\gamma_y}}\right)^m.$$

By definition of the decoupling of $(x,y) = \widetilde{\gamma_x} + \widetilde{\gamma_y} + \alpha$, we find that $\sigma \cdot \partial\widetilde{\gamma_x} = d\sigma \cdot \widetilde{\gamma_x} = d\widetilde{\gamma_x}$ for all $\sigma \in G_x$. A multiplicative analogue of Lemma 3.1.19 implies easily that $H^{\text{mul}}_{d\widetilde{\gamma_x}}$ is left fixed by G_x so that $H^{\text{mul}}_{d\widetilde{\gamma_x}}$ belongs to $k(x)$. A similar argument shows that $H^{\text{mul}}_{d\widetilde{\gamma_y}}$ belongs to $k(y)$ which concludes the proof. □

4.5 Effective construction

4.5.1 Decoupling with level lines

The evaluation of a regular fraction at a vertex of the orbit, that is, at a pair of algebraic elements in $\mathbb{K}$ might be difficult from an algorithmic point of view since this requires to compute in an algebraic extension of $\mathbb{Q}(x,y)$. This is however the cost we may have to pay in our decoupling procedure if we choose random families of 1-chains satisfying the assumptions of Theorem 4.4.12.

In this section, we show how, under mild assumption on the *distance transitivity* of the graph of the orbit, one can construct a decoupling in the orbit expressed in terms of specific 0-chains that we call *level lines*.

These level lines regroup vertices of the orbit that satisfy the same polynomial relations. Therefore, one can use symmetric functions and efficient methods from computer algebra to evaluate regular fractions on these level lines (see Appendix 4.5.2).

Definition 4.5.1. Let a be a vertex of $\mathcal{O}$. We define the *x-distance* of a to be

$$d_x(a) = \inf\{d(a,a') \,:\, a' \sim^x (x,y)\},$$

that is, the length of a shortest path in $\mathcal{O}$ from a to the clique $(x,\cdot)$.

Such a shortest path $(g_0, g_1, \ldots, g_r)$, that is, $g_r = a$, $g_0 \sim^x (x,y)$ and $d_x(a) = r$, is called an *x-geodesic* for a. Note that we have $d_x(g_i) = i$ for all $i = 0, \ldots, r$. We denote by $\mathcal{P}^x_a$ the set of 1-chains associated with x-geodesics for a as in Example 4.4.5.

The *x-level lines* $\mathcal{X}_0, \mathcal{X}_1, \ldots$ are defined by $\mathcal{X}_i = \{a \in \mathcal{O} \,:\, d_x(a) = i\}$, and we associate to the level line $\mathcal{X}_i$ the 0-chain $X_i = \sum_{a \in \mathcal{X}_i} a$. Analogously, we define the y-distance d_y, the set $\mathcal{P}^y_a$ of y-geodesics for a, the y-level lines $\mathcal{Y}_0, \mathcal{Y}_1, \ldots$, and denote by Y_i the 0-chain associated with the y-level line $\mathcal{Y}_i$.

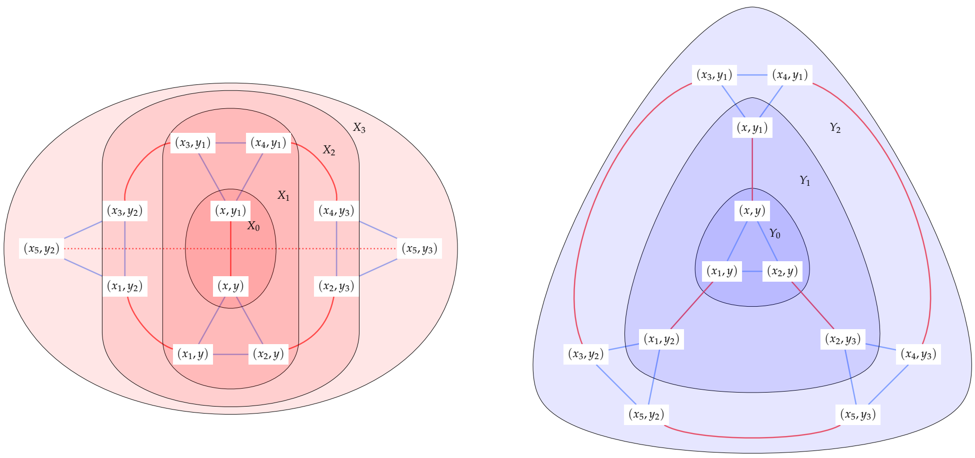


Figure 4.5.1 – The level lines for the orbit $\mathcal{O}_{12}$

The level lines can be represented graphically, as in Figure 4.5.1, in the examples of Section 4.6. The level lines and geodesics are our key tools to construct relevant collections of 1-chains satisfying the conditions of Theorem 4.4.12. First, the boundaries of a geodesic are easy to express.

Lemma 4.5.2. *Let a be a vertex of $\mathcal{O}$ and $(g_0, g_1, \ldots, g_r)$ an x-geodesic for a, then $g_i \sim^y g_{i-1}$ if and only if i is odd. Similarly, for $(g_0, g_1, \ldots, g_r)$ a y-geodesic for a, then $g_i \sim^x g_{i-1}$ if and only if i is odd.*

Proof. Let $g = (g_0, g_1, \ldots, g_r)$ be an x-geodesic of length r. Assume that there exists i such that $g_i \sim^x g_{i+1} \sim^x g_{i+2}$. By transitivity of $\sim^x$, this implies that $g_i \sim^x g_{i+2}$, contradicting the minimality of the geodesic g. Similarly, if there exists i such that $g_i \sim^y g_{i+1} \sim^y g_{i+2}$ then $g_i \sim^y g_{i+2}$, also contradicting the minimality of the geodesic.

Therefore, the adjacency types of the edges of the geodesic alternate. Finally, if $g_0 \sim^x g_1$, then this also contradicts the minimality of the geodesic because then $(x, y) \sim^x g_1$. This fixes the starting parity of the alternating adjacency types of edges of the geodesic, and thus $g_i \sim^y g_{i-1}$ if and only if i is odd. The case of a y-geodesic is symmetric. $\square$

Corollary 4.5.3. *Let a be a vertex of $\mathcal{O}$, $(g_0, g_1, \ldots, g_r)$ an x-geodesic for a and g its associated 1-chain, then $\partial^y(g) = \sum_{\substack{1 \le i \le r \\ i \text{ odd}}} g_i - g_{i-1}$. Analogously, for $(g_0, g_1, \ldots, g_r)$ a y-geodesic for a then $\partial^x(g) = \sum_{\substack{1 \le i \le r \\ i \text{ odd}}} g_i - g_{i-1}$.*

Recall from Section 4.4.1 that any graph automorphism τ of $\mathcal{O}$ acts on the vertex a of $\mathcal{O}$ coordinate-wise and that we denote this action $\tau \cdot a$. We extend the action of τ to any path $(a_1, \ldots, a_{n+1})$ as follows

$$\tau \cdot (a_1, \ldots, a_{n+1}) = (\tau \cdot a_1, \ldots, \tau \cdot a_{n+1}).$$

Note that this action is compatible with the action of graph automorphisms on 1-chains defined in Section 4.4.1. Indeed, if p is the 1-chain associated with the path $(a_1, \ldots, a_{n+1})$ as in Example 4.4.5 then $\tau \cdot p$ is the 1-chain associated with the path $\tau \cdot (a_1, \ldots, a_{n+1})$.

The following lemma shows that the geodesics and level lines satisfy some stability properties with respect to the action of elements of G_x and G_y viewed as subgroups of the group of graph automorphisms of $\mathcal{O}$.

Lemma 4.5.4. *Let σ_x be in G_x and a in $\mathcal{O}$. Then $d_x(\sigma_x \cdot a) = d_x(a)$. Moreover, if $(g_0, \ldots, g_r)$ is an x-geodesic for a, then $\sigma_x \cdot (g_0, \ldots, g_r)$ is an x-geodesic for $\sigma_x \cdot a$. Analogously, if σ_y is in G_y and a in $\mathcal{O}$, then $d_y(\sigma_y \cdot a) = d_y(a)$, and if $(g_0, \ldots, g_r)$ is a y-geodesic for a, so is $\sigma_y \cdot (g_0, \ldots, g_r)$ for $\sigma_y \cdot a$.*

Proof. Assume that $d_x(a) = r$. Then there exists an x-geodesic for a that is $(g_0, g_1, \ldots, g_r)$ with $g_r = a$. Apply the graph automorphism σ_x to each of the vertices of this path. Then $(\sigma_x \cdot g_0, \sigma_x \cdot g_1, \ldots, \sigma_x \cdot g_r)$ with $\sigma_x \cdot g_r = \sigma_x \cdot a$ is a path of the orbit. By definition, $g_0 \sim^x (x, y)$, thus $\sigma_x \cdot g_0 \sim^x (x, y)$ since x is fixed by G_x. Therefore, $d_x(\sigma_x \cdot a) \leq r = d_x(a)$. Since σ_x is an automorphism, we conclude that $d_x(\sigma_x \cdot a) = d_x(a)$. We finally deduce that $\sigma_x \cdot (g_0, \ldots, g_r)$ is an x-geodesic for $\sigma_x \cdot a$. □

This observation leads us to define two subgroups of automorphisms of the graph $\mathcal{O}$. We denote by $\mathrm{Aut}_x(\mathcal{O})$ (resp. $\mathrm{Aut}_y(\mathcal{O})$) the subgroup of graph automorphisms of $\mathcal{O}$ that preserve the x (resp. y)-distance and the adjacency types[‡]. By definition, any element τ in $\mathrm{Aut}_x(\mathcal{O})$ maps an x-geodesic for a onto an x-geodesic for $\tau \cdot a$. Moreover, a graph automorphism preserves the x-distance if and only if it induces a bijective map from $\mathcal{X}_i$ to itself for each i. Analogous results hold for $\mathrm{Aut}_y(\mathcal{O})$.

Lemma 4.5.4 implies that G_x (resp. G_y) is isomorphic to a subgroup of $\mathrm{Aut}_x(\mathcal{O})$ (resp. $\mathrm{Aut}_y(\mathcal{O})$). The benefit of the groups $\mathrm{Aut}_x(\mathcal{O})$ and $\mathrm{Aut}_y(\mathcal{O})$ is that, unlike G_x and G_y, they only depend on the graph structure of the orbit, and thus are more easily computable. Note however that not all such graph automorphisms come from a Galois automorphism (see for instance the Hadamard example in Section 4.6.2). We now state an assumption on the *distance transitivity* of the graph of the orbit.

Assumption 4.5.5. *Let a and a' be two pairs of $\mathcal{O}$. If $d_x(a) = d_x(a')$, then there exists σ_x in $\mathrm{Aut}_x(\mathcal{O})$ such that $\sigma_x \cdot a = a'$. Similarly, if $d_y(a) = d_y(a')$, then there exists σ_y in $\mathrm{Aut}_y(\mathcal{O})$ such that $\sigma_y(a) = a'$. In other words, $\mathrm{Aut}_x(\mathcal{O})$ (resp. $\mathrm{Aut}_y(\mathcal{O})$) acts transitively on $\mathcal{X}_i$ (resp. $\mathcal{Y}_i$) for all i.*

This assumption has been checked for all the finite orbit types appearing for models with steps in $\{-1, 0, 1, 2\}^2$ as well as for Hadamard and Fan-models (see the examples in 4.6). To prove that Assumption 4.5.5 is satisfied in practice, we only need to find a subgroup of $\mathrm{Aut}_x(\mathcal{O})$ that acts transitively on the orbit. This does not require to compute the full group of graph automorphisms which might be quite hard in general. However, Assumption 4.5.5 does not always hold as illustrated in the following example.

‡. One can show that this last condition is redundant with the condition on the distance preservation.

Example 4.5.6. Consider the weighted model described by the Laurent polynomial $S(x,y) = \left(x + \frac{1}{x} + y + \frac{1}{y}\right)^2$. The kernel polynomial $\widetilde{K}(x,y)$ is an irreducible polynomial of degree 4 in x and in y. Therefore, the cardinal of $\mathcal{Y}_0$ is 4 and the only right coordinate of the elements in $\mathcal{Y}_0$ is y.

Moreover, each element of $\mathcal{Y}_0$ is x-adjacent to three distinct elements in $\mathcal{Y}_1$ so the cardinality of $\mathcal{Y}_1$ is 12. Now, it is easily seen that the right coordinates of vertices in $\mathcal{Y}_0 \cup \mathcal{Y}_1$ are the roots of the polynomial $\mathrm{Res}(\widetilde{K}(X,y,1/S(x,y)), \widetilde{K}(X,y,1/S(x,y)), X)$. Since x and y are algebraically independent over $\mathbb{C}$, its irreducible factors in $\mathbb{C}(x,y)[Y]$ are

$$(Yy-1)\,,(-y+Y)\,,\left(Y^2xy + 2Y\,x^2y + Yx\,y^2 + Yx + 2Yy + xy\right)$$
$$\text{and}\ \left(Y^2xy - 2Y\,x^2y - 3Yx\,y^2 - 3Yx - 2Yy + xy\right).$$

This proves that the cardinality of the set $\mathcal{V}$ of right-coordinates of elements in $\mathcal{Y}_1$ is 5.

If Assumption 4.5.5 were true for this model then the transitive action of $\mathrm{Aut}_y(\mathcal{O})$ on $\mathcal{Y}_1$ implies that the sets $K_v = \{(u,w)\ :\ w = v \text{ and } (u,w) \in \mathcal{Y}_1\} \subset \mathcal{Y}_1$ for v in $\mathcal{V}$ are all in bijection. Indeed, K_v is equal to $\{a \in \mathcal{O}\ :\ a \sim^y (u,v)\} \cap \mathcal{Y}_1$ for some $(u,v) \in K_v$.

Therefore, as Assumption 4.5.5 provides σ_y in $\mathrm{Aut}_y(\mathcal{O})$ such that $\sigma_y \cdot (u,v) = (u',v') \in K_{v'}$, its restriction to K_v gives an embedding into $K_{v'}$, because σ_y preserves the y-adjacencies and the y-distance. By symmetry, this proves that K_v and $K_{v'}$ are in bijection. Since these sets form a partition of $\mathcal{Y}_1$, this would imply that the cardinality of $\mathcal{V}$ (5) divides the cardinality of $\mathcal{Y}_1$ (12). A contradiction. ∎

We now show that Assumption 4.5.5 is sufficient for (x,y) to admit a decoupling in terms of level lines.

Lemma 4.5.7 (Under Assumption 4.5.5)**.** *Let a and a' be two vertices with $d_x(a) = d_x(a')$. Then there is a bijection between $\mathcal{P}_a^x$ and $\mathcal{P}_{a'}^x$. Analogously, if a and a' satisfy $d_y(a) = d_y(a')$, then there is a bijection between $\mathcal{P}_a^y$ and $\mathcal{P}_{a'}^y$.*

Proof. Use Assumption 4.5.5 to produce σ_x in $\mathrm{Aut}_x(\mathcal{O})$ such that $\sigma_x(a) = a'$. This σ_x induces a bijection between $\mathcal{P}_a^x$ and $\mathcal{P}_{\sigma_x \cdot a}^x = \mathcal{P}_{a'}^x$ by Lemma 4.5.4 and the compatibility between the action of σ_x on x-geodesics and its action on the associated 1-chains. □

The following theorem gives a decoupling of (x,y) in terms of level lines.

Theorem 4.5.8 (Under Assumption 4.5.5)**.** *Define the following* 0*-chains:*

$$\gamma_x = -\frac{1}{|\mathcal{O}|}\sum_{i\geq 1}|\mathcal{X}_i| \sum_{\substack{1\leq j\leq i\\ j \text{ odd}}} \left(\frac{X_j}{|\mathcal{X}_j|} - \frac{X_{j-1}}{|\mathcal{X}_{j-1}|}\right) \quad \text{and} \quad \gamma_y = -\frac{1}{|\mathcal{O}|}\sum_{i\geq 1}|\mathcal{Y}_i| \sum_{\substack{1\leq j\leq i\\ j \text{ odd}}} \left(\frac{Y_j}{|\mathcal{Y}_j|} - \frac{Y_{j-1}}{|\mathcal{Y}_{j-1}|}\right).$$

Then $(x,y) = (\omega + \gamma_x) + \gamma_y + \alpha$ is a decoupling of (x,y) in the orbit (with $\omega = \frac{1}{|\mathcal{O}|}\sum_{a\in\mathcal{O}} a$).

Proof. Consider the two families of 1-chains $(p_a^x)_{a\in\mathcal{O}}$ and $(p_a^y)_{a\in\mathcal{O}}$ defined for a in $\mathcal{O}$ as

$$p_a^x = \frac{1}{|\mathcal{P}_a^x|}\sum_{g\in\mathcal{P}_a^x} g \quad \text{and} \quad p_a^y = \frac{1}{|\mathcal{P}_a^y|}\sum_{g\in\mathcal{P}_a^y} g.$$

For all $g=(g_0,\dots,g_r)$ in $\mathcal{P}_a^x$, we have $\partial(g)=a-g_0$ with $g_0\sim^x(x,y)$.

Then, $\varepsilon^x(\partial(g)-a+(x,y))=0$. Thus, we find by linearity that $\varepsilon^x(\partial(p_a^x)-a+(x,y))=0$. The same argument shows that $\varepsilon^y(\partial(p_a^y)-a+(x,y))=0$. Therefore, both families of 1-chains $(p_a^x)_{a\in\mathcal{O}}$ and $(p_a^y)_{a\in\mathcal{O}}$ satisfy the conditions of Theorem 4.4.12, which thus states that if we take

$$\gamma_x = -\frac{1}{|\mathcal{O}|}\sum_{a\in\mathcal{O}}\partial^y(p_a^x) \quad \text{and} \quad \gamma_y = -\frac{1}{|\mathcal{O}|}\sum_{a\in\mathcal{O}}\partial^x(p_a^y),$$

then the pair $(\omega+\gamma_x,\gamma_y)$ is a pseudo decoupling. As the geodesics are stable under the action of their respective Galois groups by Lemma 4.5.4, it is also a decoupling.

Therefore, we are left to prove that γ_x and γ_y admit the announced (pleasant) expressions. We only treat the case of γ_x, the case of y being totally symmetric.

First, note that, by Lemma 4.5.7, the cardinality of $\mathcal{P}_a^x$ (resp. $\mathcal{P}_a^y$) depend only on the x-distance (resp. y-distance) of a. For i a non-negative integer, we denote by m_i^x (resp. m_i^y) the cardinality of $\mathcal{P}_a^x$ (resp. $\mathcal{P}_a^y$) for any a such that $d_x(a)=i$ (resp. $d_y(a)=i$).

The expression of the boundary of a geodesic (Lemma 4.5.3) combined with the partition of $\mathcal{O}$ into x-level lines yields

$$\gamma_x = -\frac{1}{|\mathcal{O}|}\sum_{i\geq 0}\sum_{a\in\mathcal{X}_i}\partial^y(p_a^x) = -\frac{1}{|\mathcal{O}|}\sum_{i\geq 0}\frac{1}{m_i^x}\sum_{a\in\mathcal{X}_i}\sum_{g\in\mathcal{P}_a^x}\sum_{\substack{j \text{ odd}\\ j\leq i}}(g_j-g_{j-1}).$$

If we denote

$$S_j^i = \frac{1}{m_i^x}\sum_{a\in\mathcal{X}_i}\sum_{g\in\mathcal{P}_a^x} g_j,$$

then γ_x rewrites as

$$\gamma_x = -\frac{1}{|\mathcal{O}|}\sum_{i\geq 1}\sum_{\substack{j \text{ odd}\\ j\leq i}} S_j^i - S_{j-1}^i.$$

First, observe that, for any x-geodesic $(g_0,\dots,g_i)$, the j-th component g_j has x-distance j, so the vertices appearing in S_j^i with nonzero coefficients are in $\mathcal{X}_j$. Thus, we can write

$$S_j^i = \sum_{b\in\mathcal{X}_j}\lambda_b^{i,j} b.$$

Let σ_x be in $\mathrm{Aut}_x(\mathcal{O})$. Remind that σ_x induces a bijection on each x-level line and maps bijectively $\mathcal{P}_a^x$ and $\mathcal{P}_{\sigma_x\cdot a}^x$ for all a. Thus, we find

$$\sigma_x \cdot S^i_j = \frac{1}{m^x_i} \sum_{a \in \mathcal{X}_i} \sum_{g \in \mathcal{P}^x_a} \sigma_x \cdot g_j = \frac{1}{m^x_i} \sum_{a \in \mathcal{X}_i} \sum_{g \in \mathcal{P}^x_a} (\sigma_x \cdot g)_j = \frac{1}{m^x_i} \sum_{a \in \mathcal{X}_i} \sum_{g \in \mathcal{P}^x_{\sigma_x \cdot a}} g_j = S^i_j.$$

Under Assumption 4.5.5, the group $\mathrm{Aut}_x(\mathcal{O})$ acts transitively on $\mathcal{X}_j$. Since S^i_j is fixed by the action of $\mathrm{Aut}_x(\mathcal{O})$, one concludes easily that all the coefficients $\lambda^{i,j}_b$ are equal to some scalar λ^i_j and that $S^i_j = \lambda^i_j X_j \quad (*)$. To compute the value of λ^i_j, we recall the existence of the augmentation morphism $\varepsilon : C_0(\mathcal{O}) \to \mathbb{C}$ which associates to a 0-chain the sum of its coefficients. We apply ε to each side of $(*)$. On the one hand, $\varepsilon(S^i_j) = \sum_{a \in \mathcal{X}_i} \frac{1}{|\mathcal{P}^x_a|} \sum_{g \in \mathcal{P}^x_a} 1 = \sum_{a \in \mathcal{X}_i} 1 = |\mathcal{X}_i|$. On the other hand, $\varepsilon(\lambda^i_j X_j) = \lambda^i_j |\mathcal{X}_j|$. Therefore, we deduce $\lambda^i_j = \frac{|\mathcal{X}_i|}{|\mathcal{X}_j|}$ and the announced expression for the decoupling follows. □

Remark 4.5.9. For C a field of characteristic zero and $\widetilde{K}(x,y)$ an irreducible polynomial in $C[x,y]$, the semi-algorithm in [Buc24] tests the existence and computes, if it exists, a Galois decoupling for a given regular fraction $H(x,y)$ in $C(x,y)$. This algorithm also works when the orbit is infinite but it might not always terminate.

Buchacher's strategy consists in determining the poles on the kernel curve $\overline{E_t}$ (Section 2.1.1) of a possible decoupling pair $(F(x), G(y))$ of $H(x,y)$. This relies on the computation of the *dynamical orbit* of the poles (α, β) of $H(x,y)$, viewed as a function on the kernel curve E_t. More precisely, one takes the horizontal line $\mathbb{P}^1 \times \{\beta\}$, intersects it with E_t, then takes the vertical lines through these intersection points and determines their intersections with E_t and so on. The collection of points of E_t obtained in this way is the *dynamical orbit* of the point (α, β).

The orbit of $S(x,y)$ as defined in Section 3.1.1 coincides with the *dynamical orbit* of the generic point of $\overline{E_t}$ (see [Har77, Example 2.3.4]). Therefore, it happens that the *dynamical orbit* of certain points on $\overline{E_t}$ is finite while the *dynamical orbit* of the generic point is infinite.

Our approach relies crucially on the fact that the *dynamical orbit* of the generic point is finite and we need to compute its orbit entirely in order to use Galois theoretic arguments. Thus, the complexity of our decoupling algorithm depends on the complexity of the orbit. However, the computation of the orbit does not depend on the regular fraction $H(x,y)$.

The complexity of the algorithm of [Buc24] depends on the algebraic complexity of the poles of $H(x,y)$ and of their dynamical orbit. Hence, Buchacher's algorithm might be more efficient than ours if the algebraic complexity of the poles of $H(x,y)$ and their dynamical orbit is not too big.

Our approach generalizes [BBR21, Theorem 4.11] which is only valid for a cyclic orbit. In [DHRS18; HS21], the authors give a criterion to test the decoupling of a regular fraction $H(x,y)$ that is based on the computation of orbit-residues. These orbit-residues are essentially the sum of the residues of $H(x,y)$ along the *dynamical orbit* of a given pole of $H(x,y)$. The algorithm developed in [Buc24] relies on similar ideas. ■

To conclude, we have defined in this section a *distance-transitivity* property that is only graph-theoretic. When this property is satisfied by the orbit-type, it leads to a decoupling expressed in terms of level lines. As described in Appendix 4.5.2, the evaluation of a regular fraction on a level line is efficient from an algorithmic point of view and so is our procedure for the Galois decoupling of a regular fraction. In Section 4.6, we easily check Assumption 4.5.5 on various orbit-types and produce the associated decoupling in terms of level-lines.

4.5.2 Formal computation of decoupling with level lines

As explained in Section 4.5.1, the evaluation of a regular fraction at an arbitrary pair of elements in the orbit is expensive from a computer algebra point of view. We describe below a family of 0-chains called *symmetric chains* which are easy to evaluate on.

We will then show that the level lines introduced in Section 4.5.1 can be described explicitly in terms of these symmetric chains. Thus, under Assumption 4.5.5, Theorem 4.5.8 yields an expression of the decoupling in the orbit in terms of symmetric chains which provides a powerful implementation of the computation of the Galois decoupling of a regular fraction (see the Sage notebook).

Symmetric chains on the orbit

Definition 4.5.10. Let $P(Z)$ be a square-free polynomial in $\mathbb{C}(x,y)[Z]$. We define two finite subsets of $\mathbb{K} \times \mathbb{K}$ to be $V^1(P) = \{(u,v) \in \mathbb{K} \times \mathbb{K} : P(u) = 0 \wedge S(x,y) = S(u,v)\}$ and $V^2(P) = \{(u,v) \in \mathbb{K} \times \mathbb{K} : P(v) = 0 \wedge S(x,y) = S(u,v)\}$.

We recall here a well known corollary of the theory of symmetric polynomials (see [Lan02, Theorem 6.1]). Let $P(x)$ be a polynomial with coefficients in a field L and let $x_1, \ldots, x_n$ be its roots taken with multiplicity in some algebraic closure of L. If $H(x)$ is a rational fraction over L with denominator relatively prime to $P(x)$, then the sum $\sum_i H(x_i)$ is a well defined element of L. There are numerous effective algorithms to compute such a sum based on resultants, trace of a companion matrix, Newton formula... (see for example [BFSS06]).

We extend these methods to the computation of $s = \sum_{(u,v) \in V^1(P)} H(u,v,1/S(x,y))$ for P a square-free polynomial such that $V^1(P) \subset \mathcal{O}$ and $H(x,y)$ a regular fraction as follows. By definition of $V^1(P)$, we can rewrite s as the double sum

$$s = \sum_{u \,/\, P(u)=0} \; \sum_{v \,/\, \widetilde{K}(u,v,1/S(x,y))=0} H(u,v,1/S(x,y)).$$

Consider the sum

$$\sum_{v \,/\, \widetilde{K}(x,v,1/S(x,y))=0} H(x,v,1/S(x,y)).$$

It is a well-defined element of $k(x)$ which can be computed efficiently since it is a symmetric function on the roots of the square-free polynomial $\widetilde{K}(x,y,1/S(x,y))$. Let $\Sigma(x)$ be in $k(x)$ such that

$$\Sigma(x) = \sum_{v \,/\, \widetilde{K}(x,v,1/S(x,y))=0} H(x,v,1/S(x,y)).$$

Since the group of the orbit G acts transitively on the orbit and preserves the adjacencies, it is easily seen that, for any right coordinate of the orbit u, the sum

$$\sum_{v \,/\, \widetilde{K}(u,v,1/S(x,y))=0} H(u,v,1/S(x,y))$$

coincides with $\Sigma(u)$. Then, $s = \sum_{u \,/\, P(u)=0} \Sigma(u)$ has the desired form and can also be computed efficiently since it is a symmetric function on the roots of the square-free polynomial P. The process is symmetric for $V^2(P)$. These observations motivate the following definition.

Definition 4.5.11. A *symmetric chain* is a $\mathbb{C}$-linear combination of 0-chains of the form $\sum_{a \in V^i(P)} a$ with P a square-free polynomial such that $V^i(P) \subset \mathcal{O}$.

From the above discussion, any regular fraction $H(x,y)$ can be evaluated on a symmetric chain an efficient way.

Level lines as symmetric chains

We now motivate the choice of level lines introduced in Section 4.5.1, by showing they are symmetric chains which one can construct efficiently. We recall that the *square-free part* of a polynomial $P(Z)$ in $K[Z]$ is the product of its distinct irreducible factors and can be computed as $P(Z)/\gcd(P(Z),P'(Z))$.

Now, let P be a polynomial in $\mathbb{C}(x,y)[Z]$. Then we denote by $R_{\widetilde{K},x}(P)(Z)$ the square-free part of $\mathrm{Res}(\widetilde{K}(x,Z,1/S(x,y)),P(x),x)$ in $\mathbb{C}(x,y)[Z]$. Similarly, we define $R_{\widetilde{K},y}(P)(Z)$ to be the square-free part of $\mathrm{Res}(\widetilde{K}(Z,y,1/S(x,y)),P(y),y)$ in $\mathbb{C}(x,y)[Z]$. The following lemmas are straightforward so that we omit their proofs.

Lemma 4.5.12. *Let $P(Z)$ be a polynomial in $\mathbb{C}(x,y)[Z]$. Then,*

$$V^2(R_{\widetilde{K},x}(P)(Z)) = \{a \in \mathbb{K} \times \mathbb{K} \,:\, \exists a' \in V^1(P),\ a \sim^y a'\}$$

and

$$V^1(R_{\widetilde{K},y}(P)(Z)) = \{a \in \mathbb{K} \times \mathbb{K} \,:\, \exists a' \in V^2(P),\ a \sim^x a'\}.$$

Lemma 4.5.13. *Let i be a positive integer. Any element a of $\mathcal{X}_i$ is adjacent to some element a' of $\mathcal{X}_{i-1}$. Moreover, if i is odd then $a \sim^y a'$ and if i is even then $a \sim^x a'$.*

Now, we construct by induction a sequence of square-free polynomials $(P_j^x(Z))_j \in \mathbb{C}(x,y)[Z]$ which satisfy the equations

$$V^1(P_{2i}^x(Z)) = \mathcal{X}_{2i} \cup \mathcal{X}_{2i-1} \text{ and } V^2(P_{2i+1}^x(Z)) = \mathcal{X}_{2i+1} \cup \mathcal{X}_{2i} \text{ for all } i.$$

We set $P_0^x(Z) = Z - x$ so that $V^1(P_0^x) = \mathcal{X}_0 \subset \mathcal{O}$. Now, assume that we have constructed the polynomials $P_j^x(Z)$ for $j = 0, \dots, 2i$. By Lemma 4.5.12 and the induction hypothesis, $V^2(R_{\widetilde{K},x}(P_{2i}^x)(Z))$ is composed of all the vertices that are y-adjacent to some vertex in $\mathcal{X}_{2i} \cup \mathcal{X}_{2i-1}$. Moreover, by the induction hypothesis, $V^2(P_{2i-1}^x(Z)) = \mathcal{X}_{2i-1} \cup \mathcal{X}_{2i}$. Hence, by Lemma 4.5.13 we find that

$$V^2(R_{\widetilde{K},x}(P_{2i}^x(Z))) \setminus V^2(P_{2i-1}^x(Z)) = \mathcal{X}_{2i+1} \cup \mathcal{X}_{2i}.$$

Hence, if we define $P_{2i+1}^x(Z)$ to be $R_{\widetilde{K},x}(P_{2i}^x)(Z)$ divided by its greatest common divisor with $P_{2i-1}^x(Z)$, then $P_{2i+1}^x(Z)$ is square-free, and the above equation ensures that $V^2(P_{2i+1}^x(Z)) = \mathcal{X}_{2i+1} \cup \mathcal{X}_{2i}$. We construct $P_{2i+2}^x(Z)$ using similar arguments. Analogously, one can construct a sequence of square-free polynomials $(P_j^y(Z))_j \in \mathbb{C}(x,y)[Z]$ which satisfy

$$V^1(P_{2i}^y(Z)) = \mathcal{Y}_{2i} \cup \mathcal{Y}_{2i-1} \qquad \text{and} \qquad V^2(P_{2i+1}^y(Z)) = \mathcal{Y}_{2i+1} \cup \mathcal{Y}_{2i} \text{ for all } i,$$

starting from $P_0^y(Z) = Z - y$.

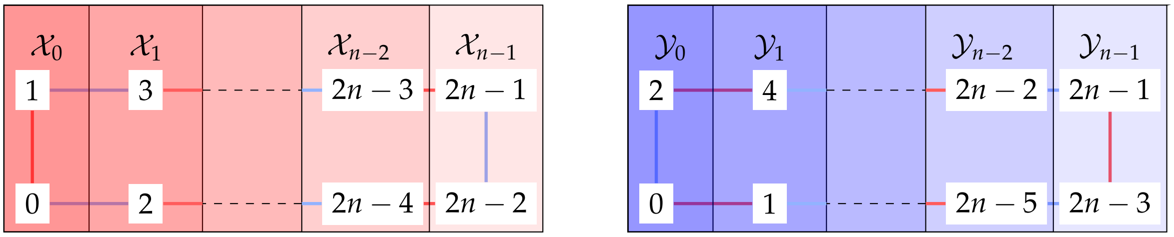
As the x-level lines are disjoint sets of vertices, the 0-chain associated with $\mathcal{X}_{i+1} \cup \mathcal{X}_i$ is just the sum $X_{i+1} + X_i$. Hence, as X_0 and all $X_{i+1} + X_i$ are symmetric chains, all the x_i are symmetric chains as well. The same argument holds for y-level lines. Note that, as expected, the coefficients of the P_i^x are actually in $k(x)$ and the coefficients of the P_i^y are in $k(y)$. By Proposition 3.1.18, one can identify $k(x)$ (resp. $k(y)$) with $\mathbb{C}(x,t)$ (resp. $\mathbb{C}(y,t)$) by identifying $1/S(x,y)$ with t, x with x and y with y so that the coefficients P_i^x (resp. P_i^y) can be considered in $\mathbb{C}(x,t)$ (resp. $\mathbb{C}(y,t)$). If the orbit is finite, the number of x or y-level lines is also finite. Thus, there is only a finite number of polynomials $P_i^x(Z)$. In the notation of Section 3.2.3, the minimal polynomial $\mu_x(Z)$ of x over $k(\mathcal{O})$ is the vanishing polynomial of the left coordinates of the orbit. It is therefore equal to the product $P_0^x(Z) \cdot P_1^x(Z) \cdot P_2^x(Z) \cdot \dots$

4.6 Examples of decoupling obstructions

In this last section, we check Assumption 4.5.5 and unroll the construction of the decoupling of the previous section for many finite orbit-types. Mainly, we treat those corresponding to walks with small steps, those corresponding to Hadamard models, those corresponding to Fan models, and finally finite orbit-types of models with steps in $\{-1,0,1,2\}^2$ (that are $\mathcal{O}_{12}$, $\mathcal{O}_{18}$ and $\widetilde{\mathcal{O}_{12}}$).

4.6.1 Orbits of cyclic type

Assume that the orbit is a cycle of size $2n$, which is the orbit-type of any small-steps model with finite orbit. The graph of the orbit looks as follows, where we have labeled vertices from 0 to $2n-1$. We represent both x-level lines and y-level lines.



Each of the x-level lines has 2 elements, so does any y-level line. The reader can check that the permutation

$$\sigma^x = (0,1)(2,3)\dots(2i,2i+1)\dots(2n-1,2n-2)$$

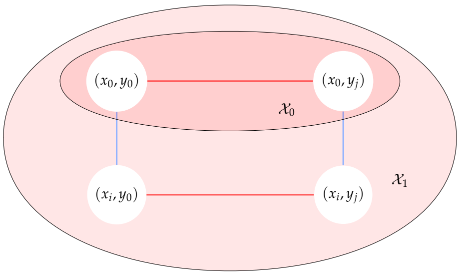
which corresponds to a horizontal reflection on the figure on the left-hand side, induces a graph automorphism of $\mathrm{Aut}_x(\mathcal{O})$, that is preserving the x-distance and the type adjacencies. Moreover, σ^x acts transitively on each x-level line. As the situation is completely symmetric for y-level lines, this proves Assumption 4.5.5 for cyclic orbits. In this section, we take the convention that the exponents on the permutation indicate which type of level lines these automorphisms stabilize. According to Theorem 4.5.8, we find:

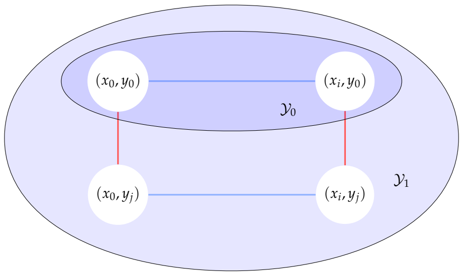
$$(x,y) = \left(\omega - \frac{1}{2n}\sum_{j\text{ odd}}(n-j)\,(x_j - x_{j-1})\right) - \left(\frac{1}{2n}\sum_{j\text{ odd}}(n-j)\,(y_j - y_{j-1})\right) + \alpha.$$

In the above equation and in the remaining of the section, we only give the explicit expressions of $\widetilde{\gamma_x}, \widetilde{\gamma_y}$ and we write them between parenthesis according to their order in the expression $(x,y) = \widetilde{\gamma_x} + \widetilde{\gamma_y} + \alpha$. The above decoupling equation corresponds to the decoupling construction obtained for small steps walks in [BBR21, Theorem 4.11].

4.6.2 Hadamard models

We recall here the form of the Hadamard orbits, computed in Proposition 3.1.14, together with annotations for the level lines.





These orbit-types are very symmetric. The x-level lines $\mathcal{X}_0$ is $\{(x,y_j) : 0 \le j \le m-1\}$ while $\mathcal{X}_1 = \{(x_i,y_j) : 0 \le j \le m-1 \text{ and } 1 \le i \le n-1\}$. Thus, $|\mathcal{X}_0| = m$ and $|\mathcal{X}_1| = (n-1)m$. It is easy to prove that any element of $\mathrm{Aut}_x(\mathcal{O})$ has the form

$$\phi^x_{\sigma,\tau} \colon (x_i,y_j) \mapsto (\sigma(x_i),\tau(y_j)),$$

for τ a permutation of the set $\{y_j : 0 \le j \le m-1\}$ and σ a permutation of $\{x_i : 0 \le i \le n-1\}$ such that $\sigma(x) = x$. An analogous description holds for the y-level lines and $\mathrm{Aut}_y(\mathcal{O})$ proving that the Hadamard models satisfy Assumption 4.5.5 and that $\mathrm{Aut}_x(\mathcal{O}) \simeq S_{n-1} \times S_m$ and $\mathrm{Aut}_y(\mathcal{O}) \simeq S_n \times S_{m-1}$. Thus, Theorem 4.5.8 gives the following decoupling:

$$\begin{aligned}(x,y) &= \frac{m(n-1)}{nm}\left(\frac{x_0}{m} - \frac{x_1}{m(n-1)}\right) + \frac{n(m-1)}{nm}\left(\frac{y_0}{n} - \frac{y_1}{n(m-1)}\right) + \omega + \alpha \\ &= \left(\frac{1}{m}x_0\right) + \left(\frac{m-1}{nm}y_0 - \frac{1}{nm}y_1\right) + \alpha = \left(\frac{1}{m}x_0\right) + \left(\frac{1}{n}y_0 - \omega\right) + \alpha,\end{aligned}$$

with $\omega = \frac{1}{mn}(y_0 + y_1)$. Note that any Hadamard model where $\deg_x \widetilde{K}(x,y) > 1$ and $\deg_y \widetilde{K}(x,y) > 1$ always contains a bicolored square, so the regular fraction xy never admits a decoupling (see Example 4.2.4).

The complete description of the groups $\mathrm{Aut}_x(\mathcal{O})$ and $\mathrm{Aut}_y(\mathcal{O})$ obtained above is particularly useful to construct examples of orbits whose graph automorphisms are not necessarily Galois automorphisms as illustrated below.

Example 4.6.1. Consider the nontrivial unweighted model defined by $S(x,y) = \left(x + \frac{1}{x}\right)\left(y^n + \frac{1}{y^n}\right)$. Then by Proposition 22 in [BBM21], the orbit has the form

$$\{x, \frac{1}{x}\} \times \{\zeta^i y, \zeta^i \frac{1}{y} \text{ for } i = 0, \dots, n-1\}$$

where ζ is a primitive n-th root of unit. Hence, the extension $k(\mathcal{O})$ equals $\mathbb{C}(x,y) = k(x,y)$. Consider the tower of field extensions $k(x) \subset k(x,y^n) \subset k(x,y)$. Since $k(x)$ coincides with $\mathbb{C}(x, y^n + \frac{1}{y^n})$ and $k(x,y^n)$ with $\mathbb{C}(x,y^n)$, the multiplicativity of the degree of a field extension yields

$$[k(\mathcal{O}) : k(x)] = [\mathbb{C}(x,y) : \mathbb{C}(x,y^n)] \times [\mathbb{C}(x,y^n) : \mathbb{C}(x, y^n + \frac{1}{y^n})] = n \times 2.$$

Indeed, since x and y are algebraically independent over $\mathbb{C}$, the element y^n is not a m-th power in $\mathbb{C}(x, y^n)$ for m dividing n. Thus, the minimal polynomial of y over the field $\mathbb{C}(x, y^n)$ is $y^n - y^n$ so that $[\mathbb{C}(x, y) : \mathbb{C}(x, y^n)]$ equals n. Moreover, since y^n does not belong to $\mathbb{C}(x, y^n + \frac{1}{y^n})$, its minimal polynomial over the later field is $y^2 - (y^n + \frac{1}{y^n})y + 1$. Thus, $G_x \subsetneq \mathrm{Aut}_x(\mathcal{O})$ because G_x is a dihedral group of size $2n$ and $\mathrm{Aut}_x(\mathcal{O})$ is S_{2n} by the above description. ∎

4.6.3 Fan models

We study a class of models derived from the ones arising in the enumeration of plane bipolar orientations (see [BFR20]). The *fan* models are derived from those introduced in [BFR20, Equation (7)] by a horizontal reflection.

Definition 4.6.2. For $i \geq 0$, define $V_i(x, y) = \sum_{0 \leq j \leq i} x^j y^{i-j}$. If $z_1, \ldots, z_p$ are complex weights, with z_p being nonzero, we define the *p-fan* to be the model with step polynomial

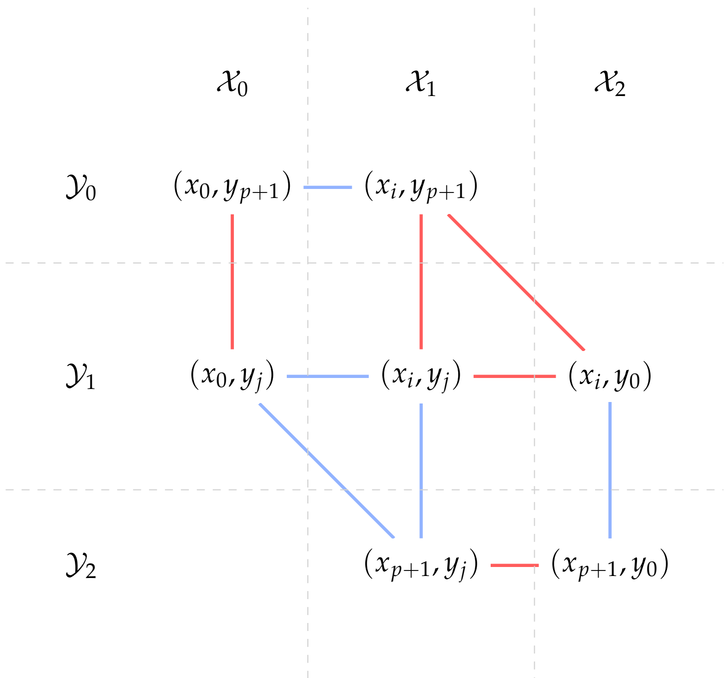
$$S(x, y) = \frac{1}{xy} + \sum_{i \leq p} z_i V_i(x, y).$$

By [BBM21, Proposition 3], the orbits of models related to one another by a reflection are isomorphic so that one can directly use the orbit computations of Proposition 4.4 in [BFR20] to compute the orbit of a p-fan.

Proposition 4.6.3. *Let $x_0, \ldots, x_p$ be defined as the roots of the equation $S(x, y) = S(x, y)$ with $x_0 = x$ and $x_{p+1} = y$. Moreover, for $0 \leq i \leq p+1$, denote $y_i = x_i$.*

In particular, $y_{p+1} = y$. Then the pairs (x_i, y_j) with $i \neq j$ form the orbit of the walk for the p-fan.

Note that all these models have small backward steps and that they all have an x/y symmetry. As a result, the orbit has size $(p+2)(p+1)$, and the cardinalities of the level lines are $|\mathcal{X}_0| = p+1$, $|\mathcal{X}_2| = p+1$ and $|\mathcal{X}_1| = p(p+1)$. The y-level lines are symmetric. Below is a depiction of this orbit type, with the indices i and j satisfy $0 < i \neq j < p+1$. Note that the orbit of the p-fan contains a bicolored square, hence no decoupling of xy is possible (see Example 4.2.4).



The groups $\mathrm{Aut}_x(\mathcal{O})$ and $\mathrm{Aut}_y(\mathcal{O})$ contain in particular the following family of automorphisms

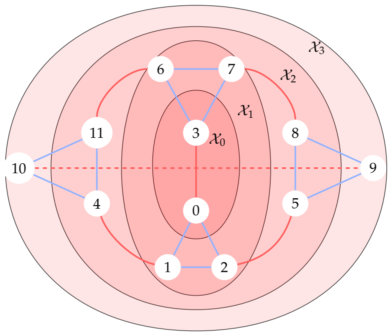
$$\phi^x_{\sigma,\tau}\colon (x_i, y_j) \mapsto (\sigma(x_i), \tau(y_j)),$$

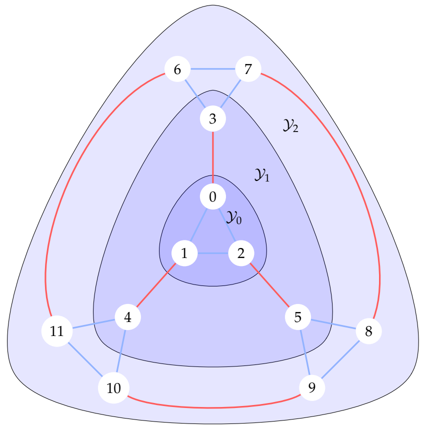
indexed by permutations σ and τ such that $\sigma(x_0) = x_0$ and $\tau(y_{p+1}) = y_{p+1}$. This family of automorphisms acts transitively on the level lines proving Assumption 4.5.5. Thus using Theorem 4.5.8 we obtain the decoupling equation of (x, y) as

$$\begin{aligned}(x,y) &= \frac{(p+1)+p(p+1)}{(p+1)(p+2)}\left(\frac{x_0}{p+1} - \frac{x_1}{p(p+1)}\right) + \frac{(p+1)+p(p+1)}{(p+1)(p+2)}\left(\frac{y_0}{p+1} - \frac{y_1}{p(p+1)}\right) + \omega + \alpha \\ &= \left(\frac{x_0}{p+1} - \frac{x_1}{p\,(p+1)\,(p+2)} + \frac{x_2}{(p+1)\,(p+2)}\right) + \left(\frac{y_0}{p+2} - \frac{y_1}{p\,(p+2)}\right) + \alpha.\end{aligned}$$

4.6.4 Orbit type $\mathcal{O}_{12}$

Below are the x and y-level lines for the orbit type $\mathcal{O}_{12}$:





Consider the following permutations of the vertices of the orbit:

$\tau^{x,y} = (1\,2)(4\,5)(6\,7)(9\,10)(8\,11)$ the vertical reflection on both sides,

$\tau^{x} = (0\,3)(1\,6)(2\,7)(4\,11)(5\,8)$ the horizontal reflection on the left-hand side,

$\tau^{y} = (0\,1\,2)(3\,4\,5)(6\,10\,8)(7\,11\,9)$ the $\frac{2\pi}{3}$ rotation on the right-hand side.

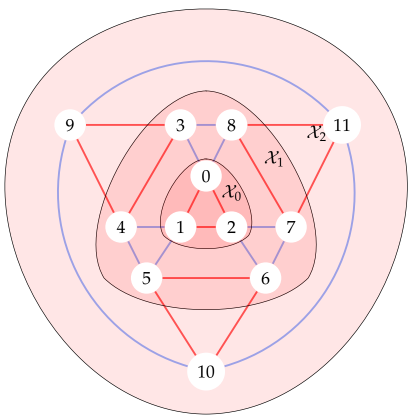
The reader can check that these automorphisms are elements of $\mathrm{Aut}_x(\mathcal{O})$ or $\mathrm{Aut}_y(\mathcal{O})$ according to their exponents and that their action on the corresponding level lines is transitive.

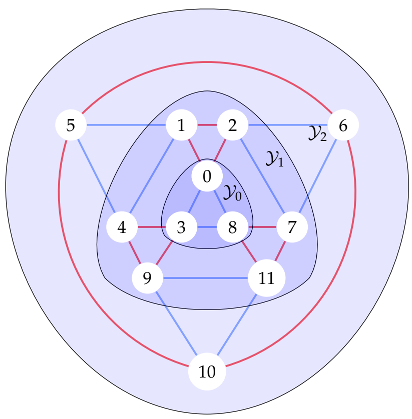
Therefore Assumption 4.5.5 holds for the orbit type $\mathcal{O}_{12}$. The cardinality of $\mathcal{O}$ is 12 and one can write $\omega = \frac{1}{12}(x_0 + x_1 + x_2 + x_3)$. Thus, according to Theorem 4.5.8, the decoupling equation is

$$\begin{aligned}(x, y) &= \frac{2}{12}\left(\frac{x_2}{4} - \frac{x_3}{2}\right) + \frac{4+4+2}{12}\left(\frac{x_0}{2} - \frac{x_1}{4}\right) + \frac{3+6}{12}\left(\frac{y_0}{3} - \frac{y_1}{3}\right) + \omega + \alpha \\ &= \left(\frac{x_0}{2} - \frac{x_1}{8} + \frac{x_2}{8}\right) + \left(\frac{y_0}{4} - \frac{y_1}{4}\right) + \alpha.\end{aligned}$$

4.6.5 Orbit type $\widetilde{\mathcal{O}_{12}}$

We represent below the x and y-level lines for the orbit type $\widetilde{\mathcal{O}_{12}}$:





We find the following automorphisms:

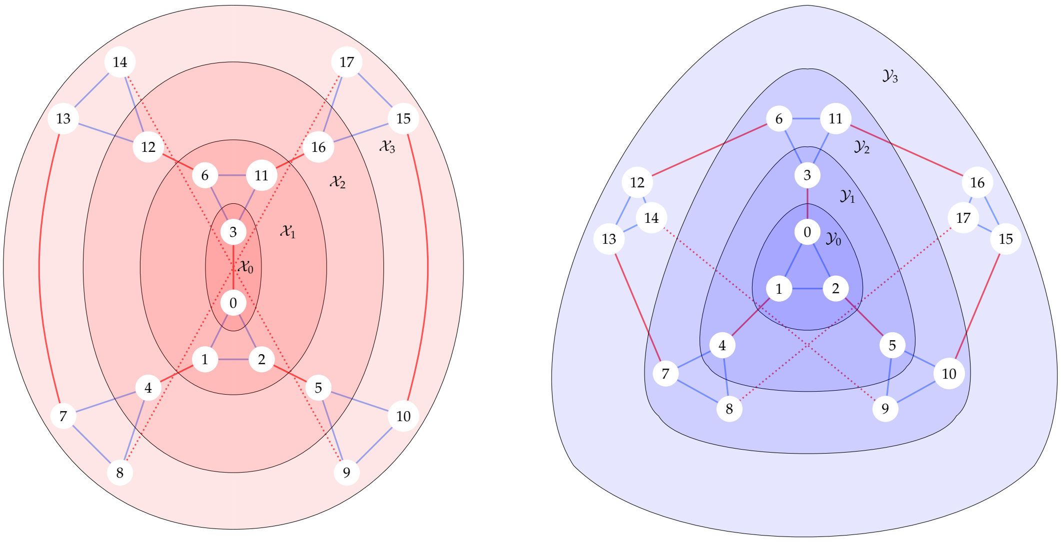
$$\tau^{x,y} = (1\,2)(3\,8)(4\,7)(5\,6)(9\,11) \qquad \text{the vertical reflection,}$$
$$\tau^{x} = (0\,1\,2)(3\,5\,7)(4\,6\,8)(9\,10\,11) \qquad \text{the } \tfrac{2\pi}{3} \text{ rotation.}$$

One can check that their action is transitive on the x-level lines. As the situation is completely symmetric for y-level lines, Assumption 4.5.5 holds for this orbit type. Thus, according to Theorem 4.5.8 and taking $\omega = \frac{1}{12}(x_0 + x_2 + x_3)$, the decoupling equation is

$$\begin{aligned}(x, y) &= \frac{6+3}{12}\left(\frac{x_0}{3} - \frac{x_1}{6}\right) + \frac{6+3}{12}\left(\frac{y_0}{3} - \frac{y_1}{6}\right) + \omega + \alpha \\ &= \left(\frac{x_0}{3} - \frac{x_1}{24} + \frac{x_2}{12}\right) + \left(\frac{y_0}{4} - \frac{y_1}{8}\right) + \alpha.\end{aligned}$$

4.6.6 Orbit type $\mathcal{O}_{18}$

We represent below the x and y-level lines for the orbit type $\mathcal{O}_{18}$.



We present some elements belonging to the groups $\mathrm{Aut}_x(\mathcal{O})$ and $\mathrm{Aut}_y(\mathcal{O})$:

$$\tau^{xy} = (1\,2)(6\,11)(4\,5)(7\,10)(8\,9)(13\,15)(14\,17)(12\,16) \qquad \text{the vertical reflection,}$$
$$\tau^{y} = (0\,1\,2)(3\,4\,5)(6\,7\,9)(8\,10\,11)(12\,13\,14)(15\,16\,17) \qquad \text{the } \tfrac{2\pi}{3} \text{ rotation for } d_y(v) \leq 2 \text{ , + rotating each "ear"}$$
$$\tau^{x}_1 = (0\,3)(1\,6)(2\,11)(4\,12)(5\,16)(7\,13)(8\,14)(9\,17)(10\,15) \qquad \text{the horizontal reflection,}$$
$$\tau^{x}_2 = (15\,17)(8\,10)(4\,5)(7\,9)(13\,14)(1\,2) \qquad \text{the pinching of the upper "arms".}$$

The reader can check that these elements act transitively on their respective level lines which proves Assumption 4.5.5 for $\mathcal{O}_{18}$. Thus, according to Theorem 4.5.8 and taking

$\omega = \frac{1}{18}(x_0 + x_1 + x_2 + x_3)$, the decoupling equation is

$$\begin{aligned}(x, y) &= \frac{8}{18}\left(\frac{x_2}{4} - \frac{x_3}{8}\right) + \frac{4+4+8}{18}\left(\frac{x_0}{2} - \frac{x_1}{4}\right) + \frac{6}{18}\left(\frac{y_2}{6} - \frac{y_3}{6}\right) + \frac{3+6+6}{18}\left(\frac{y_0}{3} - \frac{y_1}{3}\right) + \omega + \alpha \\ &= \left(\frac{x_0}{2} - \frac{x_1}{6} + \frac{x_2}{6}\right) + \left(\frac{5y_0}{18} - \frac{5y_1}{18} + \frac{y_2}{18} - \frac{y_3}{18}\right) + \alpha.\end{aligned}$$

Chapter 5

Applications to algebraicity

In Chapter 3 we studied the structure of the orbit of the walk using Galois theory. This allowed us to build tools to characterize the existence Galois invariants 3.2 and Galois decouplings of arbitrary fractions 4. We conclude that the former only exist when the orbit is finite. We then give a complete obstruction to the latter, as well as explicit way to construct them. These two notions were motivated by a strategy to prove the algebraicity of some generating functions of quadrant walks, first considered for weighted models with small steps, as seen in Section 2.2.2. From the developments of Chapters 3 and Chapter 4, the objects used in this proof, rational t-invariants and t-decouplings become viable in the setting of quadrant walks with arbitrarily large steps, and small backward steps.

The present chapter begins with a summary of the strategy to show algebraicity of the generating function of weigthed walks with small backwards steps and a finite orbit, that was illustrated on the Gessel model in Section 2.2.2. We explain how one can combine the approach of Bousquet-Mélou and Jehanne on equations with one catalytic variable [BJ06] and the notion of Galois decoupling and invariants of Chapters 3 and 4 to study the algebraicity of the generating functions for models with small backward steps.

We then give two examples of the application of this strategy in the last two sections. In the first example, we apply the strategy all the way down to prove the algebraicity of a weighted model $\mathcal{G}_\lambda$ with large forward steps. Moreover, we provide an explicit vanishing polynomial for its generating function, proving a conjecture of [BBM21]. We then use this explicit polynomial to study the asymptotics of the excursions on this model. In the second example, we consider a family of models, called the *stretched Gessel models*. We show that they all have a finite orbit, and describe it completely. In turn, we use it to construct explicit pairs of rational invariants. Moreover, we find Galois decouplings for a family of fractions $x^{a+1}y^{b+1}$. This allows us to conjecture that the generating function of walks for stretched Gessel models are algebraic for the corresponding starting points (a, b).

5.1 Algebraicity strategy

In Section 1.2.3, we showed how to derive a functional equation for the generating function $Q(x,y)$ of weighted walks in the quadrant. Its complexity arises mainly from the backwards step, that force to take into account power series derived from $Q(x,y)$ that are the *sections*, and that are the generating functions of walks terminating at a given coordinate. For weighted model whose backward steps are *small*, the number of sections that appear remains modest, and thus the functional equation is similar to the one for walks with small steps. From the reasoning made in Section 1.2.3, we obtain the following functional equation for $Q(x,y)$:

$$\begin{aligned}\widetilde{K}(x,y)Q(x,y) = x^{i_0+1}y^{j_0+1} - ty([x^{-1}]S(x,y))Q(0,y) - tx([y^{-1}]S(x,y))Q(x,0) \\ + t([x^{-1}y^{-1}]S(x,y))Q(0,0)\end{aligned} \tag{5.1.1}$$

with $\widetilde{K}(x,y) = xy(1 - tS(x,y))$, for the model is assumed to have at least one backward step in each direction.

We now see that the functional equation has the following form:

$$\widetilde{K}(x,y)Q(x,y) = xy + A(x) + B(y), \tag{5.1.2}$$

with power series $A(x) \in \mathbb{C}[x][[t]]$ and $B(y) \in \mathbb{C}[y][[t]]$ respectively equal to

$$\begin{aligned}A(x) &\stackrel{\text{def}}{=} -tx([y^{-1}]S(x,y))Q(x,0)\\ B(y) &\stackrel{\text{def}}{=} -ty([x^{-1}]S(x,y))Q(0,y) + t([x^{-1}y^{-1}]S(x,y))Q(0,0).\end{aligned}$$

The remaining of the strategy then only requires that the orbit associated to the kernel polynomial $\widetilde{K}(x,y)$ (defined in Chapter 3) is finite. The earlier example of walks with small steps (see Section 2.2.2) suggests that the finiteness of the orbit is not a superfluous assumption for proving algebraicity.

5.1.1 Decoupling and first pair of invariants

First, find a pair of t-invariants which involve $A(x)$ and $B(y)$. One way to obtain such a pair of t-invariants is by looking at (5.1.2) above.

Using the results of Chapter 4 and the finiteness of the orbit, we may decide whether the regular fraction xy admits a Galois decoupling of the form

$$xy = F(x) + G(y) + \widetilde{K}(x,y)H(x,y)$$

for $F(x) \in \mathbb{C}(x,t)$, $G(y) \in \mathbb{C}(y,t)$, and $H(x,y) \in \mathbb{C}(x,y,t)$ regular. If the fraction $H(x,y)$ has poles of bounded order at $x = 0$ and $y = 0$, this decoupling is also a t-decoupling (Definition 2.2.8). We then combine the t-decoupling with (5.1.2) to obtain the following rewriting

$$\widetilde{K}(x,y)\left(Q(x,y) - H(x,y)\right) = \left(F(x) + A(x)\right) + \left(G(y) + B(y)\right).$$

Note now that the right-hand side is of the form $U(x) + V(y)$. The series $Q(x,y)$ being a generating function of walks in the quadrant, it belongs to $\mathbb{C}[x,y][[t]]$, so it indeed has poles of bounded order at $x = 0$ and $y = 0$. By assumption on $H(x,y)$, this is also the case for the series $Q(x,y) - H(x,y)$. Therefore, the pair

$$(I_1(x), J_1(y)) = (F(x) + A(x), -G(y) - B(y)). \tag{5.1.3}$$

is a pair of t-invariants (Definition 2.2.6), and this pair involves the power series $A(x)$ and $B(y)$ as desired.

5.1.2 Galois invariants

Since the orbit is finite, Section 3.1.2 asserts that there must exists a nontrivial pair of Galois invariants $(I_2(x), J_2(y))$ (Theorem 3.2.3), and gives a way to construct them (Section 3.2.3). If the fraction $(I_2(x) - J_2(y))/\widetilde{K}(x,y)$ has poles of bounded order at $x = 0$ and $y = 0$, they are are indeed t-invariants. In the end, we get a pair of rational t-invariants

$$(I_2(x), J_2(y)). \tag{5.1.4}$$

5.1.3 Pole elimination

We now hope to be able to combine the pairs of t-invariants $(I_1(x), J_1(x))$ and $(I_2(x), J_2(y))$ through pairwise additions and multiplications in order to produce a pair of t-invariants $(I(x), J(y))$ that satisfies the conditions of Lemma 2.2.7. More precisely, we want to find a nonzero polynomial $P(X,Y) \in \mathbb{C}[[t]][X,Y]$, which allows to construct a pair of t-invariants $(I(x), J(y))$, defined as $I(x) = P(I_1(x), I_2(x))$ and $J(y) = P(J_1(x), J_2(y))$. We would like the function $(I(x) - J(y))/\widetilde{K}(x,y)$ to have no pole at $x = 0$ nor $y = 0$, so we may apply Lemma 2.2.7 to the pair of t-invariants $(I(x), J(y))$. Using Lemma 2.2.10, it is enough to construct $P(X,Y)$ so that the series $I(x)$ and $J(y)$ have no pole at $x = 0$ or $y = 0$. There is currently no guarantee that such a polynomial always exists. Hence, we do it case by case depending on the expansion in the variables x and y at 0 of the explicit pairs $(I_1(x), J_1(x))$ and $(I_2(x), J_2(y))$.

If we succeed, we thus obtain from the pair $(I_3(x), J_3(x))$ and Lemma 2.2.7 two catalytic polynomial equations on $Q(x,0)$ and $Q(0,y)$, that are $P(I_1(x), I_2(x)) = C(t)$ and $P(J_1(y), J_2(y)) = C(t)$ for some power series $C(t) \in \mathbb{C}[[t]]$. The invariants $I(x)$, $J(y)$, and the coefficients of P are polynomials in the coefficients of $A(x)$ and $B(x)$, thus the equations are actually of the form

$$P_1(A(x), x, t, C(t), F_1(t), F_2(t), \dots) = 0 \tag{5.1.5}$$

$$P_2(B(y), y, t, C(t), F_1(t), F_2(t), \dots) = 0. \tag{5.1.6}$$

If Equations (5.1.5) and (5.1.6) are well founded (see Section 2.2.2), then Theorem 2.2.1 allows us to conclude that the series $A(x)$ and $B(y)$ are algebraic over $\mathbb{C}(x,t)$ and $\mathbb{C}(y,t)$ respectively, and therefore that $Q(x,y)$ is algebraic over $\mathbb{C}(x,y,t)$. Moreover, this method being explicit, it theoretically permits to produce explicit vanishing polynomials for the series $A(x)$ and $B(y)$, and thus finally for $Q(x,y)$.

5.1.4 Discussion

For unweighted small steps models, the results of [BM10; BK10; KR12; DHRS20; MM14; BBR21] show that the generating function is algebraic in the variables x and y if and only if the model admits some non-trivial Galois invariants and xy has a Galois decoupling. For weighted models with small steps, [DR19, Corollary 4.2] and [BBR21, Theorem 4.6 and Theorem 4.11] imply that the existence of non-trivial Galois invariants and of a Galois decoupling pair for xy yield the algebraicity of the generating functions. We conjecture that the reverse implication is also true for walks with small backwards steps, yielding an equivalence which should also be valid in the large steps case. The general strategy explained above is summarized in Figure 5.1.1 and motivates the above conjecture. It is the first attempt at finding uniform proofs for the algebraicity of generating functions of large steps models.

The strategy detailed above is entirely algorithmic, except for checking that Galois invariants and decoupling are indeed t-invariants and t-decoupling, and that combining the two pairs of invariants allows us to find a pair of invariants satisfying 2.2.7, and finally that this last pair of invariants yields polynomial equations in one catalytic variable satisfying the conditions of Theorem 2.2.1. Nonetheless, we think that this could be made constructive, in particular via a generalization of the notion of weak invariants [BBR21, Section 5.2] to the large steps framework.

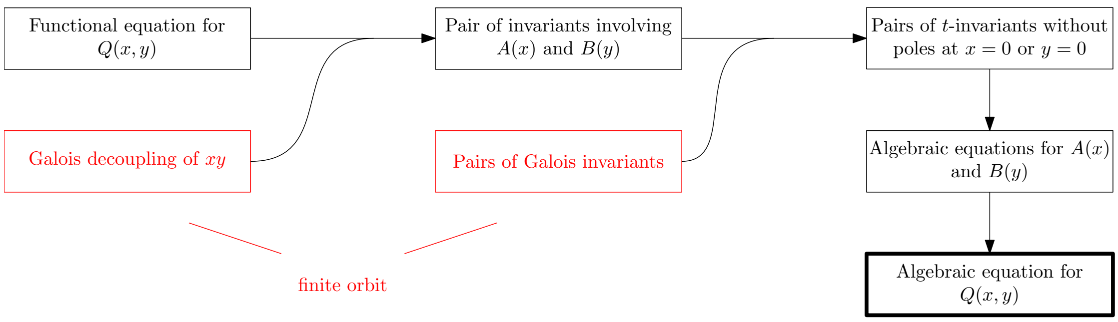


Figure 5.1.1 – Summary of the strategy for proving algebraicity of generating functions of walks with small backwards steps

5.2 The model $\mathcal{G}_\lambda$

We consider the weighted model

$$\mathcal{G}_\lambda = \{(-1,-1),(0,1),(1,-1),(2,1),((1,0),\lambda)\}$$

together with its step polynomial $S(x,y) = \frac{1}{xy} + y + \frac{x}{y} + x^2y + \lambda x$, and kernel polynomial $\widetilde{K}(x,y) = xy - t(1 + xy^2 + x^2 + x^3y^2 + \lambda x^2 y)$. The weight λ is a nonzero complex number. The goal in this section is to give a derivation of the algebraicity proof of the generating function $Q(x,y)$ of walks in the quadrant based on this weighted model.

5.2.1 Functional equation

Following the general argument of 5.1, the functional equation for $Q(x,y)$ is

$$\widetilde{K}(x,y)Q(x,y) = xy - t(x^2+1)Q(x,0) - tQ(0,y) + tQ(0,0).$$

This equation being of the shape

$$\widetilde{K}(x,y)Q(x,y) = xy + A(x) + B(y),$$

with $A(x) = -t(x^2+1)Q(x,0) + tQ(0,0)$ and $B(y) = -tQ(0,y)$, one may attempt to apply the strategy of [BBR21] to prove the algebraicity of $Q(x,y)$.

5.2.2 Initial invariants

Decoupling and first pair of invariants

One can check that xy admits the following t-decoupling:

$$xy = -\frac{3\lambda x^2 t - \lambda t - 4x}{4t(x^2+1)} + \frac{-\lambda y - 4}{4y} - \frac{\widetilde{K}(x,y)}{(x^2+1)yt}.$$

Combining this identity with the functional equation, one obtains the following pair of t-invariants:

$$(I_1(x), J_1(y)) = \left(\frac{3\lambda t\, x^2 - \lambda t - 4x}{-4t\, x^2 - 4t} - t\left(x^2+1\right)Q(x,0) + tQ(0,0)\,, tQ(0,y) + \frac{\lambda y + 4}{4y}\right).$$

Galois invariants

The pair $(I_2(x), J_2(y))$ below is a pair of t-invariants:

$$(I_2(x), J_2(y)) = \left(\frac{\left(-\lambda^2 x^3 - x^4 - x^6 + x^2 + 1\right)t^2 - x^2\lambda\left(x^2-1\right)t + x^3}{t^2 x\left(x^2+1\right)^2}, \frac{-t\, y^4 + \lambda t y + y^3 + t}{y^2 t}\right).$$

5.2.3 Pole elimination and algebraicity

As we now have two pairs of t-invariants $P_1 = (I_1(x), J_1(y))$ and $P_2 = (I_2(x), J_2(y))$, we perform some algebraic combinations between them in order to eliminate their poles. To lighten notation, we write the component-wise operations on the pairs P_i of t-invariants. Computations can be checked in the joint Maple worksheet.

Consider the Taylor expansions of the first coordinates:

$$I_1(x) = \frac{\lambda}{4} + O(x)\,,$$
$$I_2(x) = x^{-1} + O(x)\,.$$

Out of these two pairs of t-invariants, we first produce a third pair of t-invariants without a pole at $x = 0$ as follows:

$$P_3 = (I_3, J_3) \overset{\text{def}}{=} P_2 \left(P_1 - \frac{\lambda}{4} \right).$$

The first coordinates of the pairs P_1 and P_3 do not have a pole at $x = 0$. The Taylor expansion of their second coordinates $J_1(y)$ and $J_3(y)$ at $y = 0$ is as follows:

$$J_3(y) = y^{-3} + (tQ(0,0) + \lambda)\, y^{-2} + t\left(Q(0,0)\,\lambda + \frac{\partial^2 Q}{\partial y^2}(0,0) \right) y^{-1} + O(y^0),$$
$$J_1(y) = y^{-1} + O(y^0).$$

In order to produce a pair of t-invariants satisfying the assumption of the Invariant Lemma, we need to combine P_1 and P_3 in order to eliminate the pole at $y = 0$. Note that, since the first coordinates of P_1 and P_3 have no pole at zero, the first coordinate of any sum or product between these two pairs have no pole at $x = 0$. Using the simple pole at $y = 0$ of J_3, we produce a new pair P_4 whose coordinates have no pole at $x = 0$ nor $y = 0$ by setting

$$P_4 = (I_4, J_4) \overset{\text{def}}{=} P_3 - P_1^3 + \left(2tQ(0,0) - \frac{\lambda}{4} \right) P_1^2 + \left(2t\frac{\partial^2 Q}{\partial y^2}(0,0) - t^2 Q(0,0)^2 + \frac{5\lambda^2}{16} \right) P_1.$$

By Lemma 2.2.10, the pair P_4 is a pair of trivial t-invariants. Therefore, the Invariant Lemma 2.2.7 yields the existence of a series $C(t)$ in $\mathbb{C}((t))$ such that $I_4(x) = C(t)$ and $J_4(y) = C(t)$.

The value of $C(t)$ can be deduced from the values of $Q(0,y)$ and its derivatives at 0 by looking at the Taylor expansion of $J_4(y)$ at $y = 0$. The verification that the polynomial equations $I_4(x) = C(t)$ and $J_4(y) = C(t)$ are well-founded is done in the Maple worksheet. We only give here the form of the well-founded equation for $F(y) \overset{\text{def}}{=} Q(0,y)$:

$$\begin{aligned} F(y) = 1 + t\Big(& t^2 y F(y) \left(\Delta^{(1)} F(y)\right)^2 + \lambda t F(y)\, \Delta^{(1)} F(y) + t \left(\Delta^{(1)} F(y)\right)^2 \\ & + 2tF(y)\, \Delta^{(2)} F(y) + yF(y) + \lambda \Delta^{(2)} F(y) + 2\Delta^{(3)} F(y) \Big). \end{aligned} \tag{5.2.1}$$

Theorem 2.2.1 with $\mathbb{L} = \mathbb{Q}(\lambda)$ implies that the generating function of the weighted model $\mathcal{G}_\lambda$ is algebraic over $\mathbb{Q}(\lambda)(x, y, t)$. Moreover, one can show that, at any step of our reasoning, one may have taken the weight λ to be zero. In particular, the generating function of the model $\mathcal{G}_0$ is algebraic. Thus, the excursion generating functions $Q(0,0)$ of the reverse models of $\mathcal{G}_0$ and $\mathcal{G}_1$ (obtained by the map $(i,j) \mapsto (-i,-j)$) are algebraic over $\mathbb{Q}(t)$, which was conjectured in [BBM21].

5.2.4 Explicit polynomial equation for $Q(0,0)$

We start from the functional equation (5.2.1) obtained for $Q(0,y)$, because it is the simplest of the two. Besides $F(y)$, there are three unknown functions: $F(0)$ (the excursions series), $F'(0)$ and $F''(0)$. The above equation can hence be rewritten as

$$P(F(y), F(0), F'(0), F''(0), t, y) = 0, \tag{5.2.2}$$

with $P(x_0, x_1, x_2, x_3, t, y)$ a polynomial with coefficients in $\mathbb{Q}(\lambda)$.

The method of Bousquet-Mélou and Jehanne consists in constructing more equations from (5.2.2). For that purpose, we search for fractional power series* y_i's that are solutions of (5.2.2) and of the following equation

$$(\partial_{x_0} P)\,(F(y), F(0), F'(0), F''(0), t, y) = 0. \tag{5.2.3}$$

Then the paper [BJ06] points out that any such solution is also a solution of the following equation

$$(\partial_y P)\,(F(y), F(0), F'(0), F''(0), t, y) = 0. \tag{5.2.4}$$

Moreover, these solutions are double roots of $D(F(0), F'(0), F''(0), t, y)$, the discriminant of P with respect to x_0 [BJ06, Theorem 14]. If there are enough fractional power series y_i's (at least the number of unknown functions), then the result of [BJ06] provides "enough" independent polynomial equations $P_i(x_0, x_1, x_2)$ relating the unknown functions (here $F(0)$, $F'(0)$ and $F''(0)$) so that the dimension of the polynomial ideal generated by the P_i's is zero. This shows that one can eliminate the unknown series between these multivariate polynomial equations to find a one variable polynomial equation for each of the unknown series.

Let y be a solution to (5.2.3). Eliminating $F''(0)$ between (5.2.3) and (5.2.4), one finds a first equation between y and $F(y)$:

$$\begin{aligned} -2F(y)t\,y^4 + F(0)^2\,t^2 y_i - 4F(0)F(y)\,t^2 y + 3y\,t^2 F(y)^2 - F(0)\lambda t y \\ + F(y)\lambda t y + F(y)\,y^3 - 2F'(0)ty - y^3 - 4tF(0) + 4F(y)t = 0. \end{aligned} \tag{5.2.5}$$

Now, eliminating $F(y)$ between (5.2.5) and (5.2.2), and removing the trivially nonzero factors, we obtain the following polynomial equation satisfied by y:

$$2t\,y^4 - y^3 + \lambda t y + 2t = 0. \tag{5.2.6}$$

Using Newton polygon's method, we find that, among the four roots of the irreducible polynomial above, exactly three are fractional power series y_1, y_2 and y_3 that are not formal power series. The last root, denoted y_0, is a Laurent series with a simple pole at $t = 0$. Moreover, (5.2.6) yields

$$t = \frac{y_0}{2{y_0}^4 + \lambda y_0 + 2},$$

*. A fractional power series is an element of $\mathbb{C}[\![t^{1/d}]\!]$ for some positive integer d.

so that $\mathbb{Q}(\lambda, t) \subset \mathbb{Q}(\lambda, y_0)$. Replacing t by the above expression in (5.2.6) and factoring by $y - y_0$, we obtain the minimal polynomial $M(y_0, y)$ satisfied by the series y_1, y_2, y_3 over $\mathbb{Q}(\lambda, y_0)$ as:

$$M(y_0, y) = 2{y_0}^3 y^3 - {y_0}^2 \lambda y - y_0 \lambda y^2 - 2{y_0}^2 - 2y_0 y - 2y^2. \tag{5.2.7}$$

This polynomial of degree 3 is irreducible over the field $\mathbb{Q}(\lambda, y_0) \subset \mathbb{Q}(\lambda)((t))$ because otherwise one of the series y_i's would belong to $\mathbb{Q}(\lambda, y_0)$ which is impossible since the y_i's are not Laurent series in t. Since $\mathbb{Q}(\lambda, y_0, F(0), F'(0), F''(0)) \subset \mathbb{Q}(\lambda)((t))$, the same argument shows that $M(y_0, y)$ remains irreducible over the field $\mathbb{Q}(\lambda, y_0, F(0), F'(0), F''(0))$.

Now, since the y_i's are double roots of $D(F(0), F'(0), F''(0), t(y_0), y)$, the polynomial $M(y_0, y)^2$ must divide $D(F(0), F'(0), F''(0), t(y_0), y)$ so that the remainder $R(y)$ in the euclidian division of $D(F(0), F'(0), F''(0), t(y_0), y)$ by $M(y_0, y)^2$ should be identically zero. The polynomial $R(y)$ has degree at most 5 (the discriminant has degree 12 and $M(y_0, y)^2$ has degree 6), and we write it as

$$R(y) = e_1 + e_2 y + e_3 y^2 + e_4 y^3 + e_5 y^4 + e_6 y^5$$

with e_i a polynomial in y_0, $F(0)$, $F'(0)$ and $F''(0)$. Hence, each of its coefficients gives an equation $e_i = 0$ on the unknown functions in terms of y_0. We first eliminate $F''(0)$ between e_1 and e_2 which yields an equation e_7 between y_0, $F(0)$, $F'(0)$. We get another such equation e_8 by eliminating $F''(0)$ between e_1 and e_3. Finally, eliminating $F'(0)$ between e_7 and e_8 yields an equation e_9 over $\mathbb{Q}(\lambda)$ between y_0 and $F(0)$. The polynomial defining the equation e_9 factors into 6 factors. Among these 6 factors, there are two nontrivial algebraic equations for $F(0)$. To decide which of these factors is a polynomial equation for $F(0)$, we compute the first terms of the t-expansion $F(0) = Q(0, 0, t)$ (which is easy from the functional equation for $Q(x, y)$) and of $y_0(t)$ (thanks to the Newton method) and we plug these approximations in the two factors of e_9. One finds that $F(0)$ is algebraic of degree 8 over $\mathbb{Q}(\lambda)(y_0)$. One eliminates y_0 thanks to its functional equation and, thanks to Maple, one verifies that $F(0)$ is algebraic of degree 32 over $\mathbb{Q}(t)$ (see the Maple worksheet). This gives the following result:

Proposition 5.2.1. *The series $Q(0, 0)$ is algebraic of degree 8 over $\mathbb{Q}(\lambda)(y_0)$ (for any λ). Hence, as y_0 has degree 4 over $\mathbb{Q}(\lambda)(t)$, we conclude that $Q(0, 0)$ is an algebraic series of degree 32 over $\mathbb{Q}(\lambda)(t)$.*

We note that any step of our procedure remains valid if one specializes λ to 0 and 1 so that the excursion series $Q(0, 0)$ of the models $\mathcal{G}_0$ and $\mathcal{G}_1$ remains algebraic of degree 32 over $\mathbb{Q}(\lambda)(y_0)$.

5.2.5 Asymptotics

In [DW15], Denisov and Wachtel propose a method to compute the asymptotic behaviour for the number of excursions when the model is *aperiodic* (see [BBM21, Theorem 3.11] for details). This result gives that the number of excursions for the model $\mathcal{G}_\lambda$ with

$\lambda \neq 0$ is asymptoptically $\kappa\mu^n n^{-5/2}$. The model $\mathcal{G}_0$ is not aperiodic. [BBM21, Theorem 3.11] gives the value $(\frac{3}{16})^{3/4}$ of the radius of convergence of the generating function counting excursions for $\mathcal{G}_0$. However, it only gives upper and lower bounds for the asymptotic of $\mathcal{G}_0$. Using the explicit vanishing polynomial $P_1(t,y)$ for the generating function $Q(0,0) \in \mathbb{Q}[\![t]\!]$ of excursions on the model $\mathcal{G}_0$ over $\mathbb{Q}(t)$, we follow the method exposed in [FS09] to compute precisely the asymptotics of its coefficients.

We remark that, in the case $\lambda = 0$, the coefficients of P_1 belong to $\mathbb{Q}[t^4]$. Hence, we may write $Q(0,0) = F(t^4)$ for some $F \in \mathbb{Q}[\![t]\!]$. Writing $P(t^4, y) \overset{\text{def}}{=} P_1(t,y)$, we have that $F(x)$ satisfies the equation $P(t, F(t)) = 0$. We thus focus on the asymptotics of the coefficients of the power series $F(t)$. We follow the method of singularity analysis detailed in [FS09, p. VII.7.1]. Thus, we first locate the dominant singularity of $F(t)$, and then compute the Newton polygon at this singularity to determine its nature.

Proposition 5.2.2. *Let $\alpha_n = q_{4n}^{0,0}$ be the n-th coefficient of $F(t)$, that is, the number of excursions on the model $\mathcal{G}_0$ of length $4n$. Then, one has*

$$\alpha_n \sim \kappa \left(\tfrac{16}{3}\right)^{3n} n^{-5/2} \text{ with } \kappa \approx 0.07021064809.$$

Proof. We refer the reader to the joint Maple worksheet *AsymptoticG0.mw* for the computation's details. We start from the polynomial equation $P(t, F(t)) = 0$ satisfied by $F(t)$ to perform some singularity analysis.

We first compute the *exceptional set* ξ of P, which provides a set of potential candidates for the dominant singularity. It is the set of roots of the discriminant $R(t) \overset{\text{def}}{=} \text{Res}(P(t,y), \partial_y P(t,y), y)$. From Pringsheim's Theorem [FS09, Th. IV.6], the radius of convergence $\rho = (\frac{3}{16})^3$ of $F(t)$ must be a dominant singularity of $F(t)$. Moreover, we check that no other element of ξ has modulus ρ, thus ρ is the only dominant singularity of $F(t)$ (see the joint Maple worksheet for the computations).

We now determine the nature of $F(t)$ at this singularity. Computing the Puiseux expansions of the roots of $P(t, Z)$ at $t = \rho$, one sees that there are 16 branches that have an algebraic singularity at ρ with an expansion of the form $a + b(1 - \frac{t}{\rho}) + c(1 - \frac{t}{\rho})^{3/2} + O((1 - \frac{t}{\rho})^2)$, while the other ones do not have a singularity at this point. Therefore, from Darboux's theorem [Mel21, Proposition 2.11], one has $q_{4n}^{0,0} \sim \frac{c}{\Gamma(-\frac{3}{2})} \left(\frac{16}{3}\right)^{3n} n^{-5/2}$ where c is the coefficient of $(1 - \frac{t}{\rho})^{3/2}$ in the Puiseux expansion of $F(x)$. As $\Gamma(-\frac{3}{2}) \geq 0$, we know that c must be positive. But there is only one among the 16 constants c that is greater than 0, and Maple gives $c \approx 0.1659268448$, which allows us to conclude. ☐

5.3 The models $\mathcal{H}_n$ (stretched Gessel models)

In the classification of models with small steps, four of them were proved algebraic. Among them, the so-called *Gessel Model*, given by the Laurent polynomial

$$S(x,y) = (1 + 1/y)/x + (1 + y)x.$$

It was a notoriously difficult model to study, and the first known proof of algebraicity of its full generating function used heavy computer algebra (see [BK10]). Among other proofs of this result, one relied on the general strategy developped in [BBR21], which is the one presented in 2.2.2. It is noteworthy that no bijective proof of this result yet exists.

Figure 5.3.1 – The stretched Gessel models

In a private communication, Mireille Bousquet-Mélou suggested that we explore with our tools a new family of large steps models $(\mathcal{H}_n)_n$ which she expected to have a finite orbit for every non-negative integer n. These models are obtained from the Gessel model by stretching the two rightmost steps (so that we may call them the *stretched Gessel models*). More precisely, they are defined through the following Laurent polynomial

$$H_n(x,y) = (1+1/y)/x + (1+y)x^n = (1+y)(1/x/y + x^n),$$

and presented in Figure 5.3.1. We show in this section that many walks based on these models, with different starting points, give an algebraic power series $Q(x,y)$.

5.3.1 Orbit

Here, the study of this general family of models requires some study of the Galois groups G_x and G_y to compute decouplings and Galois invariants.

Lemma 5.3.1. *Let $u, v, u', v' \in \mathbb{K} = \overline{\mathbb{C}(x,y)}$.*

(i) $(u,v) \sim (u,v')$ if and only if $v' = v$ or $v' = c(u,v) \stackrel{\text{def}}{=} v^{-1}u^{-n-1}$.

(ii) $(u,v) \sim (u',v)$ if and only if $\lambda = \frac{u'}{u}$ is a solution of the equation

$$\lambda(\lambda^n - 1) = (\lambda - 1)c(u,v).$$

Proof. The polynomial $\widetilde{K}(x,y) = xy(1 - tH_n(x,y))$ has respective degrees $\deg_x \widetilde{K}(x,y) = n+1$ and $\deg_y \widetilde{K}(x,y) = 2$, hence we find 2 solutions v' for (i), and $n+1$ solutions u' for (ii). The verification that $v' = c(u,v)$ works in (i) is straightforward, hence we focus on (ii). Starting from $(u,v) \sim (u',v)$, we derive

$$\begin{aligned} & H_n(u,v) = H_n(u',v) \\ \iff & \frac{1}{uv} + u^n = \frac{1}{u'v} + u'^n \\ \iff & u'^n - u^n = \frac{u'-u}{uu'v} \\ \iff & \lambda^n - 1 = \frac{\lambda - 1}{\lambda uv} u^{-n} \\ \iff & \lambda(\lambda^n - 1) = (\lambda - 1)v^{-1}u^{-n-1}. \end{aligned}$$

□

As the kernel polynomial $\widetilde{K}(x,y)$ has degree 2 in the variable y, the extension $k(x,y)/k(x)$ is Galois, with Galois group generated by the involution σ_x. Its expression in $k(x)(y)$ is determined by its action on y, which is given by

$$y^{\sigma_x} = x^{-n-1}y^{-1} = c(x,y).$$

By the embedding extension theorem, the $k(x)$-algebra homomorphism $\sigma_x : k(x,y) \to \mathbb{K}$ can be extended to a $k(x)$-algebra homomorphism (also denoted σ_x) $\sigma_x : \mathbb{K} \to \mathbb{K}$ (see Proposition 2.1.3).

We now describe the orbit of the model $\mathcal{H}_n$. Let $1, \lambda_1, \ldots, \lambda_n$ be the $n+1$ distinct roots in $\mathbb{K}$ of the polynomial

$$Z(Z^n - 1) = (Z-1)y^{\sigma_x}.$$

By applying σ_x to the coefficients of the above polynomial, and since $\sigma_x^2 = 1$ in $k(x,y)$, one can also note that $1, \lambda_1^{\sigma_x}, \ldots, \lambda_n^{\sigma_x}$ are the $n+1$ distinct roots of the polynomial

$$Z(Z^n - 1) = (Z-1)y.$$

Having established these notations, we may now describe the orbit of the model $\mathcal{H}_n$.

Proposition 5.3.2. *For $n \geq 1$, the model $\mathcal{H}_n$ has a finite orbit, of cardinality $2(n+1)^2$, as illustrated in Figure 5.3.2.*

Proof. We describe the pairs of the orbit that are respectively at x-distance 0, 1, 2 and 3, or equivalently, belonging to $\mathcal{X}_0$, $\mathcal{X}_1$, $\mathcal{X}_2$ and $\mathcal{X}_3$ (see Section 4.5.1). In particular, we show that the x-diameter of the orbit is 3. By Lemma 5.3.1, the pairs in $\mathcal{X}_0$ are (x,y) and (x, y^{σ_x}), so we only need to compute the x-geodesics starting from these two points, and show that they have finite length and that there is a finite number of them. We start from the x-geodesics starting at (x,y). Recall that $c(u,v)$ was defined in Lemma 5.3.1.

$\mathcal{X}_1$. Consider $u \in \mathbb{K}$ such that $(u,y) \sim^y (x,y)$. Then by Lemma 5.3.1, $u = \lambda_i x$ for some $i \in \{1, \ldots, n\}$.

$\mathcal{X}_2$. Consider $v \in \mathbb{K}$ and i in $\{1, \ldots, n\}$ with $(\lambda_i\, x, v) \sim^x (\lambda_i x, y)$. By Lemma 5.3.1, one must have
$$v = c(\lambda_i x, y) = \lambda_i^{-n-1}\, x^{-n-1}\, y = \lambda_i^{-n-1} y^{\sigma_x}.$$

$\mathcal{X}_3$. Consider $u \in \mathbb{K}$ and $i \in \{1, \ldots, n\}$ such that $(u, \lambda_i^{-n-1}\, y^{\sigma_x}) \sim^y (\lambda_i\, x, \lambda_i^{-n-1}\, y^{\sigma_x})$. By Lemma 5.3.1, one must have that $u = \mu \lambda_i\, x$, with μ a root of the polynomial $Z(Z^n - 1) - (Z-1)c(\lambda_i\, x, \lambda_i^{-n-1}\, y^{\sigma_x})$. We compute
$$\begin{aligned} c(\lambda_i\, x, \lambda_i^{-n-1}\, y^{\sigma_x}) &= \lambda_i^{-n-1}\, x^{-n-1}\, \lambda_i^{n+1}\, y\, x^{n+1} \\ &= y. \end{aligned}$$

Hence, by definition, one must have $\mu = \lambda_j^{\sigma_x}$ for some $j \in \{1, \ldots, n\}$.

Similarly, we computes through the same arguments the x-geodesics starting from (x, y^{σ_x}). We obtain Figure 5.3.2, and conclude to the finiteness of the orbit, since no pair has x-distance 4. The x-cliques are then of the form $(x, \cdot)$, $(\lambda_i\, x, \cdot)$, $(\lambda_j^{\sigma_x}\, x, \cdot)$ and $(\lambda_i\, \lambda_j^{\sigma_x}\, x, \cdot)$, hence the orbit has size $2 \times (1 + n + n + n^2) = 2(n+1)^2$. □

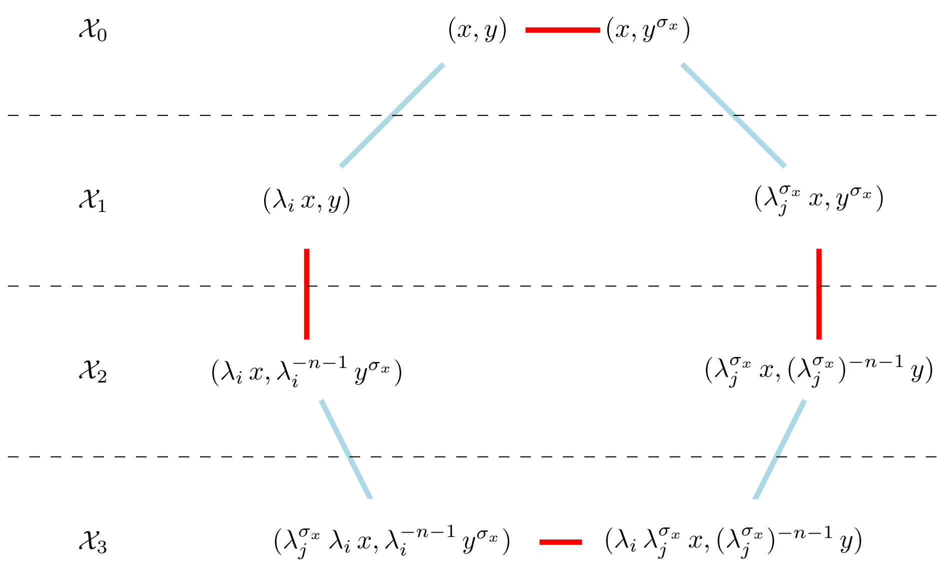


Figure 5.3.2 – The orbit of the model $\mathcal{H}_n$

From the above description of the orbit, we are in particular able to compute the minimal polynomial of the right coordinates
$$\mu_y(Z) = \prod_i (Z - y^{\sigma_x} \lambda_i^{-n-1}) \cdot \prod_i (Z - y(\lambda_i^{\sigma_x})^{-n-1}).$$

Indeed, let λ be one of the roots of the polynomial $Z(Z^n - 1) - (Z-1)y$, so that $\lambda \in \{1, \lambda_1, \ldots, \lambda_n\}$. Then if we write $z = \lambda^{-n-1}y$, we derive

$$\begin{aligned}
&\lambda(\lambda^n - 1) = (\lambda - 1)y \\
\Rightarrow &\lambda^{n+1} + y = \lambda \cdot (y+1) \\
\Rightarrow &(\lambda^{n+1} + y)^{n+1} = \lambda^{n+1} \cdot (y+1)^{n+1} \\
\Rightarrow &(y/z + y)^{n+1} = y/z \cdot (y+1)^{n+1} \\
\Rightarrow &(z+1)^{n+1} - \frac{(y+1)^{n+1}}{y^n} z^n = 0.
\end{aligned}$$

Since $\lambda_i^{n+1} = \lambda_i \cdot (y+1) - y$ and the λ_i are distinct, then so are the λ_i^{n+1}, and so are the roots of the polynomial $(Z+1)^{n+1} - \frac{(y+1)^{n+1}}{y^n} Z^n$ of degree $n+1$. Following the same technique, we find that the minimal polynomial of the right coordinates of the orbit can be expressed as

$$\mu_y(Z) = \left((1+Z)^{n+1} - \frac{(1+y)^{n+1}}{y^n} Z^n \right) \cdot \left((1+Z)^{n+1} - \left(\frac{(1+y)^{n+1}}{y^n} \right)^{\sigma_x} Z^n \right).$$

5.3.2 Initial invariants

We now construct the first two pairs of invariants in order to prove algebraicity. In contrast with the study of the single model $\mathcal{G}_\lambda$ of Section 5.2, where a single orbit is considered and where the implemented algorithms can be directly applied, we treat here the whole family of stretched Gessel models $\mathcal{H}_n$, depending on the parameter n.

Galois invariants

Since the orbit is finite by Proposition 5.3.2, Theorem 3.2.3 ensures that there must exist a nontrivial pair of Galois invariants. Below, we show two ways to construct them.

The first one consists in exploiting the minimal polynomial of the right coordinates of the orbit $\mu_y(Z)$. From Section 3.2.3, we know that a generator of the field of Galois invariants can be found among the coefficients of this polynomial. More precisely, any nonconstant coefficient generates k_{inv}. Let

$$f(y) \stackrel{\text{def}}{=} \frac{(1+y)^{n+1}}{y^n}.$$

By expanding $\mu_y(Z)$, we have that

$$\mu_y(Z) = (1+Z)^{2(n+1)} - (f(y) + f(y^{\sigma_x}))Z^n(1+Z)^{n+1} + (f(y)f(y^{\sigma_x}))Z^{2n}.$$

We have that $[Z^{2n+1}]\mu_y(Z) = 2(n+1) - (f(y) + f(y^{\sigma_x}))$. We also know that

$$y^{\sigma_x} = x^{-n-1}y^{-1} \tag{5.3.1}$$

and also that

$$(1+y)^{\sigma_x} = 1 + x^{-n-1}y^{-1} = H_n(x,y)x^{-n}(1+y)^{-1}. \tag{5.3.2}$$

Hence, we have that

$$f(y) + f(y^{\sigma_x}) = \frac{(1+y)^{n+1}}{y^n} + \frac{H_n(x,y)^{n+1}y^n}{(1+y)^{n+1}}.$$

We now see that the Galois invariant $f(y) + f(y^{\sigma_x})$ does not belong to k (for instance, $y = 0$ is a pole of this fraction), therefore

$$k_{\text{inv}} = k\left(f(y) + f(y^{\sigma_x})\right).$$

We thus let the first pair of Galois invariants be $(I_1(x), J_1(y))$, with

$$J_1(y) = \frac{(y+1)^{n+1}}{y^n} + \frac{y^n}{(1+y)^{n+1}t^{n+1}}.$$

Once this simple invariant noticed, we can a posteriori explain how we could have found it. Starting from a fraction $f(y)$, we know that $f(y) + f(y^{\sigma_x})$ must be a Galois invariant, for σ_x has order 2. If $f(y^{\sigma_x})$ has an explicit expression in terms of y, then it is easy to check if $f(y) + f(y^{\sigma_x})$ is a nontrivial Galois invariant. Consider then the equations (5.3.1) and (5.3.2). If we want to form such an $f(y)$, we may simply kill the contribution of factors in x in both y^{σ_x} and $(y+1)^{\sigma_x}$. This dictates that we must take $f(y) = \frac{(1+y)^{n+1}}{y^n}$, and we again see that $f(y) + f(y^{\sigma_x})$ is not an element of k. This gives an alternative proof of the finiteness of the orbit.

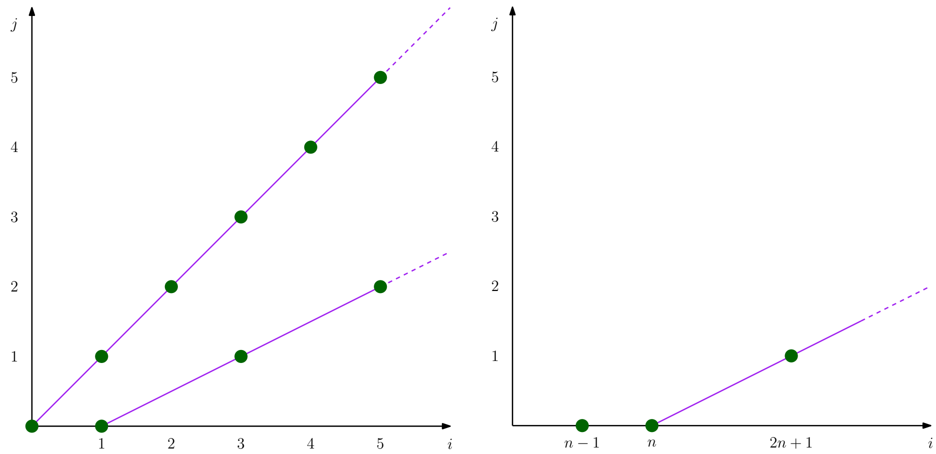
Note that while the expression of $f(y) + f(y^{\sigma_x})$ in terms of y is simple, the expression of $f(y) + f(y^{\sigma_x})$ in terms of x is more complicated, and does not seems to admit a nice closed form for a general n.

Decoupling and other pair of invariants

Now that the orbit is guaranteed to be finite, it is natural to apply the criteria of Chapter 4 to test whether the fraction xy admits a decoupling. This is the case for the Gessel model $\mathcal{H}_1$. However, we checked that it is not the case for the generalized versions $\mathcal{H}_n$ when $4 \geq n \geq 2$, so it seems unlikely that the generating function counting walks starting from the point $(0,0)$ for any of these larger models is algebraic.

This is however not the end of the story, for we may investigate other starting points (i,j) that could lead to an algebraic power series. This amounts to testing the decoupling of the fraction $x^{i+1}y^{j+1}$. Proposition 7.3 in [BBR21] implies that, for $\mathcal{H}_1$, the fractions of the form $x^a y^b$ with $a,b \geq 1$ that admit a decoupling are precisely those satisfying $(a,b) = (k,k)$ or $(a,b) = (2k,k)$ for some $k \geq 0$. This includes walks starting at $(0,0)$ (corresponding to the fraction xy), but also other starting points, lying on two lines (Figure 5.3.3, left).

This result leads us to look for such points, trying to recover a similar pattern for the stretched Gessel models $\mathcal{H}_n$ when $n \geq 2$. To this end, we investigate systematically



(a) $\mathcal{H}_1$ (b) $\mathcal{H}_n$ for $n \geq 2$ (conjectured)

Figure 5.3.3 – The algebraic starting points for the Gessel models

the Galois decoupling of monomials $x^a y^b$ with exponents (a, b) near the origin, which leads us to the following conjecture.

Conjecture 5.3.3. For $n \geq 2$, the fraction $x^a y^b$ admits a t-decoupling with respect to the model $\mathcal{H}_n$ if and only if $(a, b) = (n, 1)$ or $(a, b) = ((n+1)k, k)$ for some k.

Note that there is a discrepancy between the case $n = 1$ and the other cases. Mainly, for $n \geq 2$, there is only one conjectured line of algebraic starting points (the line $((n+1)k - 1, k - 1))$) while the diagonal $(k+1, k+1)$ only exists for $\mathcal{H}_1$, being replaced by the single point $(n-1, 0)$ for $n \geq 2$.

It is easy to prove the positive part of the conjecture, meaning that for all n and the listed (a, b) the fraction $x^a y^b$ indeed admits a Galois decoupling with respect to $\mathcal{H}_n$.

1. For $(a, b) = (n, 1)$, we need to show that $x^n y$ admits a decoupling. This is the case, thanks to the following identity

$$x^n y = -\frac{1}{x} + \frac{y}{t(y+1)} - \frac{\widetilde{K}(x, y)}{tx(y+1)}.$$

2. For $(a, b) = ((n+1)k, k)$, we show that $x^a y^b$ admits a decoupling. Let $k \geq 1$, and consider the element

$$f_k \stackrel{\text{def}}{=} y^k + (y^k)^{\sigma_x} = y^k + \frac{1}{y^k x^{k(n+1)}}.$$

By construction, f_k is fixed by σ_x, so the Galois correspondence implies that $f_k = F_k(x, 1/H_n(x, y))$ for some $F_k(x, t) \in \mathbb{Q}(x, t)$. It follows that

$$x^{k(n+1)} y^k = F_k(x, 1/H_n(x, y)) x^{k(n+1)} - \frac{1}{y^k},$$

and thus that $(F_k(x)x^{k(n+1)}, -\frac{1}{y^k})$ is a Galois decoupling of $x^{k(n+1)}y^k$. Note that as for the Galois invariants, the expression for $F_k(x)$ in terms of x and t seems complicated.

The proof of the conjecture seems doable using the results of Chapter 4, for they give complete obstructions to the decoupling of regular fractions. Below, we draw the connection between the existence of a Galois decoupling of $x^a y^b$ and the fact that some fraction can be expressed in terms of another one.

Proposition 5.3.4. *Let $a, b \in \mathbb{N}_{>0}$ be positive integers, and set*

$$c \overset{\text{def}}{=} a - b \cdot (n+1).$$

If the fraction $x^a y^b$ admits a Galois decoupling, then $c = 0$ or there is a fraction $g \in \mathbb{C}(Z)$ with

$$f\left(Z\frac{Z^n - 1}{Z - 1}\right) = \frac{Z^c - 1}{Z^a - 1}.$$

Proof. Since we showed above that $x^a y^b$ admits a Galois decoupling when $c = 0$ or $(a, b) = (n, 1)$ (in which case, $Z\frac{Z^n-1}{Z-1} = -\frac{Z^a-1}{Z^c-1}$), we may assume that $c \neq 0$ and $(a, b) \neq (n, 1)$. We apply the criterion of Proposition 4.2.7, meaning that $H(x, y) = x^a y^b$ admits a Galois decoupling if and only if $H_\alpha = 0$ for all α canceling decoupled fractions. In turn, from Proposition 4.2.3, the α's that need to be tested are exactly those induced by bicolored cycles. It is not hard to see from Figure 5.3.2 that every bicolored cycle of $\mathcal{O}$ has the form of Figure 5.3.2, parametrized by two integers i and j.

We thus evaluate $H(x, y)$ on such cycle. To lighten notation, we write $\xi_j \overset{\text{def}}{=} (\lambda_j)^{\sigma_x}$. The condition $H_\alpha = 0$ for all such α is equivalent for all i, j to the following equation

$$(1 - \lambda_i^a)x^a y^b + (1 - \xi_j^a)\lambda_i^c x^a (y^{\sigma_x})^b = (1 - \xi_j^a)x^a (y^{\sigma_x})^b + (1 - \lambda_i^a)\xi_j^c x^a y^b.$$

In turn, this may be rewritten further into

$$(1 - \lambda_i^a)(1 - \xi_j^c)y^b = (1 - \xi_j^a)(1 - \lambda_i^c)(y^{\sigma_x})^b. \qquad (E_{i,j})$$

For $n \geq 2$, our goal is to show that there is no other solution than $(a, b) = (n, 1)$.

Note that the λ_i and ξ_i are transcendental over $\mathbb{C}$ since they generate the transcendental functions y and y^{σ_x} over $\mathbb{C}$ (for instance, $\lambda_i(\lambda_i^n - 1)/(\lambda_i - 1) = y^{\sigma_x}$). In particular, λ_j and ξ_j are not roots of units. Therefore, and since $a, c \neq 0$, both sides of $(E_{j,j})$ are nonzero. We may thus divide $(E_{i,j})$ by $(E_{j,j})$ for all i, j, to obtain the following equations for all i, j

$$\frac{1 - \lambda_i^a}{1 - \lambda_j^a} = \frac{1 - \lambda_i^c}{1 - \lambda_j^c}. \qquad (5.3.3)$$

and then

$$\frac{1 - \lambda_i^c}{1 - \lambda_i^a} = \frac{1 - \lambda_1^c}{1 - \lambda_1^a} \qquad (5.3.4)$$

for all i.

Recall that the λ_i's are roots of the polynomial

$$Z\frac{Z^n - 1}{Z - 1} - y^{\sigma_x}.$$

Therefore, the extension $\mathbb{C}(y^{\sigma_x}, \lambda_1, \dots, \lambda_n)/\mathbb{C}(y^{\sigma_x})$ is Galois. By (5.3.4), the element $\frac{1-\lambda_1^c}{1-\lambda_1^a}$ is fixed by the Galois group. Therefore, by the Galois correspondence, there must exist a fraction $f(Z) \in \mathbb{C}(Z)$ such that

$$\frac{1 - \lambda_1^c}{1 - \lambda_1^a} = f(y^{\sigma_x}).$$

But

$$y^{\sigma_x} = \lambda_1 \frac{\lambda_1^n - 1}{\lambda_1 - 1},$$

therefore one has

$$f\left(\lambda_1 \frac{\lambda_1^n - 1}{\lambda_1 - 1}\right) = \frac{\lambda_1^c - 1}{\lambda_1^a - 1}.$$

Since λ_1 is transcendental over $\mathbb{C}$, we may as well consider it to be an indeterminate Z, hence the claimed factorization. □

It now remains to show that this last obstruction, which is purely field theoretic corresponds to $(a, b) = (n, 1)$. We hope that the arithmetic of the algebraic function field $\mathbb{C}(Z)/\mathbb{C}$ will allow us to prove this fact.

5.3.3 Pole elimination and algebraicity

From the explicit pairs of invariants and decouplings obtained earlier, we explicitly proved algebraicity of the models $\mathcal{H}_n$ for the starting points $(n-1, 0)$ and $(n, 0)$, with $n = 2, 3$. Since the y-part from the Galois invariants and Galois decoupling is quite explicit, we hope to give a general argument to prove that for starting points $(a, b) = (n-1, 0)$ and $(a, b) = ((n+1)k - 1, k - 1)$, the generating function $Q(x, y)$ is algebraic. Unfortunately, the converse might be tricky. Even if we prove Conjecture 5.3.3, there is no general argument for showing the non algebraicity.

In an upcoming paper, we plan to prove the conjecture, along with a proof that for all $n \geq 2$ and all values of (a, b) such that $x^{a+1}y^{b+1}$ admits a decoupling, the generating function of quadrant walks on $\mathcal{H}_n$ starting from (a, b) is algebraic.

Chapter 6

Decoupling with an infinite group : the case of walks with interacting boundaries

The chapters 3, 4 and 5 were centered around the case of a finite orbit, and algebraicity proofs. In particular, there always existed Galois invariants, and the problem of determining whether a function admitted a Galois decoupling was shown to be solved. In the present chapter, we will see the counterpart where the group is infinite, for weighted models with small steps. Our presentation thus illustrates more the Section 2.1 of Chapter 2. We will see how to use the results of difference Galois theory to reduce the problem of algebraicity of a generating function to a “decoupling” problem in a function field. In turn, we will tackle this problem using different techniques than in Chapter 4, namely pole propagation.

This chapter is centered around the study of *walks with interacting boundaries*. This problem, first introduced in [TOR14], consists in counting weighted walks in a cone on some model $\mathcal{S}$ while accounting for the number of contacts with the boundaries (the *interactions*). For quadrant walks, which we focus on, this amounts to count the number of contacts of the walks with the axes $i = 0$ and $j = 0$. The study of such problems leads to the study of the *phase transitions* of the model [Jan15]. More precisely, one parametrizes the tendency of the model to stick to the boundary $i = 0$ (resp. $j = 0$) via the *Boltzmann weight* $b \in \mathbb{R}^+$ (resp. $a \in \mathbb{R}^+$). The qualitative behaviour of the system changes depending on the Boltzmann weights, thus defining its phases. Counting the walks relative to the interaction statistics is done through functional equations that generalize those presented in Section 1.2.3. As a result, some of the techniques developed for counting walks in the quadrant may be applied to this setting. For small steps models, the techniques of Chapter 2 allowed authors to classify the generating function of some models for some Boltzmann weights [BOX21; BOR19], or even to solve them to get the full phase transition diagram such as in [TOR14].

We focus on the so-called *weighted models of genus zero* which correspond to the five sets of steps below. They correspond to the case where the curve $\overline{E_t}$ has genus 0, see

Section 2.1.3. For these sets of steps, we give the full classification of the algebraic-

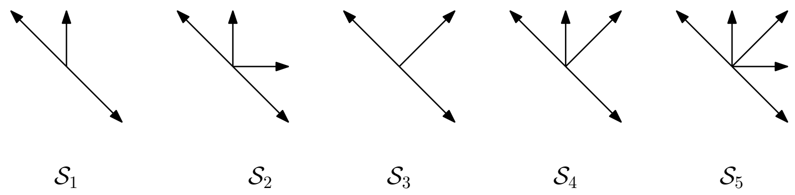


Figure 6.0.1 – The five models of genus 0

differential nature of generating function $Q(x,y)$ of weighted walks with interacting boundaries with respect to the variables x and y. Recall that for every weighting $(d_v)_{v\in\mathcal{S}}$ attached to the steps and Boltzmann weights a and b, the generating function $Q(x,y)$ is defined as

$$Q(x,y) = \sum_{w \text{ walk}} \text{weight}(w) x^i y^j t^n,$$

where for w a walk, (i,j) are its ending coordinates, n is its length, and weight(w) is a monomial depending on the weights $(d_v)_{v\in\mathcal{S}}$, a and b (defined in Section 6.1.1).

To do this, we adapt the strategy of [DHRS20] that was used to classify the generating function of weighted walks based upon on the sets of steps of Figure 6.1.1. This amounts to study the rational solutions of some functional equation (a *q-difference equation*), whose coefficients depend on the parameters a, b and $(d_v)_{v\in\mathcal{S}}$. This q-difference equation will be obtained in the manner described in Section 2.1.4.

In Section 6.2 we exploit the symmetries of the q-difference equation to reduce the classification of its solutions to the study of two decoupling problems. We develop criteria based on pole propagation to test the existence of rational solutions to decoupling equations of a special form. These criterias reduce the classification of $Q(x,y)$ to relations between the weights, and they provide a concise way to explore the space of parameters, to either prove the nonexistence of such decouplings or to find solutions. These methods call for eventual generalizations (Section 7.2.3).

Using this technique, we ultimately obtain the form of the following theorem:

Theorem (Theorem 6.5.8, Section 6.5.3). *For any weighted genus 0 model, the generating function $Q(x,y)$ of weighted walks in the quadrant with interacting boundaries has the following nature in the variables x and y:*

1. *For the sets of steps $\mathcal{S}_1$ or $\mathcal{S}_2$ and Boltzmann weights satisfying $a+b=ab$, the generating function $Q(x,y)$ is* ***rational*** *with specializations $Q(x,0)$ and $Q(0,y)$ respectively equal to*

$$Q(x,0) = \frac{1}{1 - x\dfrac{ad_{1,0}t + abd_{1,-1}d_{0,1}t^2}{1-abd_{1,-1}d_{-1,1}t^2}}, \quad Q(0,y) = \frac{1}{1 - y\dfrac{bd_{0,1}t + abd_{-1,1}d_{1,0}t^2}{1-abd_{1,-1}d_{-1,1}t^2}}.$$

2. *For the set of steps $\mathcal{S}_3$ and Boltzmann weights $a=b=2$, the generating function $Q(x,y)$ is* ***algebraic*** *of degree at most 4, with specializations $Q(x,0)$ and $Q(0,y)$ respectively*

equal to

$$Q(x,0) = \frac{1}{\sqrt{1 - x^2 \frac{4d_{1,1}d_{1,-1}t^2}{1-4d_{1,-1}d_{-1,1}t^2}}}, \quad Q(0,y) = \frac{1}{\sqrt{1 - y^2 \frac{4d_{1,1}d_{-1,1}t^2}{1-4d_{1,-1}d_{-1,1}t^2}}}.$$

3. *In every other case, the series $Q(x,y)$ is **non x-D-algebraic nor y-D-algebraic** (meaning $Q(x,y)$ satisfies no polynomial differential equation in x nor in y for any choice of x, y, t and weighting $(d_v)_{v\in\mathcal{S}}$, a and b).*

In [DHRS20] where the Boltzmann weights a and b are both equal to one, the generating functions of the models were found to be all non x-D-algebraic nor y-D-algebraic. The addition of the Boltzmann weights a and b allows us to find algebraic models.

Organization of the chapter

In Section 6.1, we recall standard definitions and facts in the study of quadrant walks, mainly their statistics, the weighting associated to a walk given a weighting $((d_v)_{v\in\mathcal{S}}, a, b)$, the generating function $Q(x,y)$ of such walks, and the algebraic-differential classification of bivariate power series. We then focus on the five sets of steps of Figure 6.1.1, for which the kernel curve has genus 0. As a result, the kernel curve admits a rational parametrization $(x(s), y(s))$, for which we will recall basic facts, among which the existence of an automorphism $\sigma(s) \stackrel{\text{def}}{=} qs$ for some real number q which is not a root of unit. We then proceed to evaluate the functional equation for $Q(x,y)$ on this curve, this way obtaining two independent functional equations (q-difference equations) on the functions $\widetilde{F}(s) = Q(x(s),0)$ and $\widetilde{G}(s) = Q(0,y(s))$. We then compare the algebraic-differential properties of these two functions with those of $Q(x,y)$, thus reducing to the study of $\widetilde{F}(s)$ and $\widetilde{G}(s)$ through these q-difference equations.

In Section 6.2, we thus devise the strategy for determining the algebraic-differential nature of $\widetilde{F}(s)$ and $\widetilde{G}(s)$. The analytic properties of q-difference equations being rigid enough, a theorem by Ishizaki allows us to reduce the classification to two *decoupling problems*, introduced in Lemma 6.2.2, one said homogeneous, the other one inhomogeneous. The classification will then go as follows. For most set of steps and weighting (see Section 6.5.2 for the one exception), and depending on the existence of solutions to these decoupling equations, either we will be in the case of Lemma 6.2.5, and then the generating function will not be D-algebraic in x and y, either we will be in the case of Lemma 6.2.7, and then we will be able to give explicit algebraic solutions for $Q(x,y)$.

Section 6.3 is thus devoted to the study of the rational solutions to decoupling equations of the form $\gamma_1(x(s),y(s))f(x(s)) + \gamma_2(x(s),y(s))g(y(s)) + c = 0$ for all $s \in \mathbb{P}^1$, with fractions γ_1, γ_2 and c depending on the weights $(d_v)_{v\in\mathcal{S}}$, a, b and t. Through a process called *pole propagation*, we will see that the existence of rational solutions is conditioned to the relative positions of some particular points of $\mathbb{P}^1$ with respect to the action of σ. More explicitly, the relative position between two points is defined as the unique integer $n = \delta(P,Q)$ such that $\sigma^n P = Q$, which we call the *σ-distance*. In the end, we extract necessary conditions for the existence of rational solutions to the decoupling problem, based on the values $\delta(P,Q)$ for $(P,Q) \in \mathcal{L}^- \times \mathcal{L}^+$, for some finite sets $\mathcal{L}^-$ and $\mathcal{L}^+$.

Section 6.4 gives a way to compute this σ-distance, based on the fact that we can define valuations on the coordinates of the points that are considered in this chapter (they are the orbits of points in $\mathcal{L}^-$ and $\mathcal{L}^+$). These valuations evolve with the action of σ in a deterministic way, allowing to effectively compute the σ-distance between two points given a fixed weighting. We extend this algorithm to find algebraic relations between the weights that guarantee a certain σ-distance.

In Section 6.5, we finally exploit the σ-distance computation of Section 6.4 along with the criteria determined in Section 6.3 to treat all the cases. In the end, we obtain the classification in the form of Theorem 6.5.8.

The Section 7.2 discusses various questions left at the end of the present chapter, related to alternative proofs for the algebraic cases, the phase transitions of the models, and the general study of these inhomogeneous decoupling equations.

6.1 Quadrant walks and q-difference equations

6.1.1 Quadrant walks with interacting boundaries

In order to introduce the generating function for walks with interacting boundaries, we briefly reintroduce the objects covered in Chapters 1 and 2.

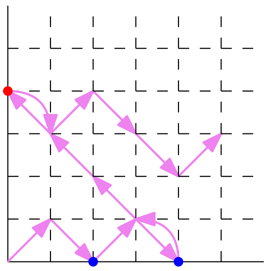


Figure 6.1.1 – A walk in the quadrant with set of steps $\mathcal{S} = \{(-1,1),(1,-1),(1,1)\}$, using 13 steps, 2 contacts with the x-axis and 1 contact with the y-axis.

Consider a finite subset $\mathcal{S}$ of vectors in $\mathbb{Z}^2 \setminus \{(0,0)\}$. When studying quadrant walks based on $\mathcal{S}$, several statistics are included. For a walk w of n steps, we still consider the coordinate (i,j) of the last point w visits, as well as the number n_v of occurrences of the step v for all v in $\mathcal{S}$.

The *interacting boundaries* qualification refers to the addition of the following statistics. We now consider the number of *contacts* (also called interactions) of w with the axes, which we briefly saw when considering Dyck path, in Chapter 1. Recall that a contact with the x-axis occurs at each $i \geq 1$ such that $\sum_{j \leq i} v_j$ is zero. Thus, performing twice the step $(1,0)$ starting from $(0,0)$ accounts for two contacts with the x-axis, despite the walk remaining on the x-axis. The number of contacts with the x-axis (resp. y-axis) is denoted by n_x (resp. n_y).

Probability distribution on weighted walks

In combinatorics, it is quite common to associate weights to objects, depending on their statistics, often for probabilistic purposes. Below, we define the weight of quadrant walks that we consider, based on the interactions.

For each $v \in \mathcal{S}$ is given $d_v > 0$ the real positive weight associated with the step v. Moreover, we are also given positive real numbers $a > 0$ (resp. $b > 0$) the *Boltzmann weight* associated with the x-axis (resp. y-axis). The weight of the walk w is then defined as the monomial

$$\text{weight}(w) \stackrel{\text{def}}{=} \left(\prod_{v\in\mathcal{S}} d_v^{n_v}\right) a^{n_x} b^{n_y}.$$

This definition of weight is correlated with an associated probability distribution on walks of length n, defined so that the probability of a walk w is proportional to its weight, i.e.

$$\mathbb{P}_n(w) \stackrel{\text{def}}{=} \frac{\text{weight}(w)}{\sum\limits_{w' \text{ walk of length } n} \text{weight}(w')}.$$

The weighting "influences" the general shape of the walk, for instance a bigger value for d_v increases the probability of performing the step v, or a small value of b favors walks that have fewer contacts with the y-axis. The above heuristics can be made precised through limit properties of the probability $\mathbb{P}_n$, that define *phases* (see Section 7.2.1).

We call a *weighted model* of quadrant walks with interacting boundaries a set of steps $\mathcal{S}$ together with a *weighting*, i.e. positive real weights $(d_v)_{v\in\mathcal{S}}$, and Boltzmann weights $a > 0$ and $b > 0$. Given a weighted model, we write

$$\mathbb{F} \stackrel{\text{def}}{=} \mathbb{Q}((d_v)_{v\in\mathcal{S}}, a, b)$$

for the subfield of $\mathbb{C}$ generated by the weights.

Generating function and functional equation

Given a weighted model $(\mathcal{S}, (d_v)_{v\in\mathcal{S}}, a, b)$, the generating function of quadrant walks on this weighted model is defined as

$$Q(x,y) \stackrel{\text{def}}{=} \sum_{w \text{ walk}} \text{weight}(w) x^i y^j t^n = \sum_{w \text{ walk}} \left(\prod_{v\in\mathcal{S}} d_v^{n_v}\right) a^{n_x} b^{n_y} x^i y^j t^n.$$

Since there is a finite number of quadrant walks of length n and that the walks always terminate in the first quadrant, the generating function $Q(x,y)$ belongs to $\mathbb{F}[x,y][[t]]$ (recall that $R[[t]]$ denotes the ring of formal power series in the variable t with coefficients in the ring R). Note that it is harmless to have the d_v, a and b as real numbers for the exact counting of walks with regards to the statistics n_v, n_x and n_y. As the transcendence degree of $\mathbb{C}$ over $\mathbb{Q}$ is infinite, one may choose algebraically independent weights d_v, a and b over $\mathbb{Q}$, and still perform coefficient extraction to get these statistics since the coefficient $[t^n]Q(x,y)$ belongs to $\mathbb{Q}[(d_v)_{v\in\mathcal{S}}, a, b, x, y]$. This is why we directly consider the generating function of weighted walks, only having x, y and t as variables.

The generating function is characterized through a functional equation. In Theorem 6 of [BOR19] the authors derive the following explicit functional equation for the series $Q(x,y)$ when the steps of the model $\mathcal{S}$ are small (that is $\mathcal{S} \subseteq \{-1,0,1\}^2$):

$$\begin{aligned} K(x,y)Q(x,y) = \frac{xy}{ab} + x\left(y - \frac{y}{a} - tA_{-1}(x)\right)Q(x,0) \\ + y\left(x - \frac{x}{b} - tB_{-1}(y)\right)Q(0,y) - \left(\frac{xy}{ab}(1-a)(1-b) - t\varepsilon\right)Q(0,0). \end{aligned} \tag{6.1.1}$$

This functional equation generalizes those found in the quadrant walks literature for the study of weighted models independently of the interaction statistics. Recall that the polynomial $K(x,y) \overset{\text{def}}{=} xy(1-tS(x,y))$ is called the *kernel*, where $S(x,y) \overset{\text{def}}{=} \sum_{(i,j)\in\mathcal{S}} d_{i,j}x^iy^j$ encodes the set of steps as a Laurent polynomial. Here, the fractions $A_i(x)$ and $B_i(y)$ are then defined as $A_i(x) \overset{\text{def}}{=} [y^i]S(x,y)$ and $B_j(y) \overset{\text{def}}{=} [x^j]S(x,y)$. Finally, according to the notation of [BOR19], the variable ε is set to 1 if the step $(-1,-1)$ is an element of $\mathcal{S}$, and $\varepsilon = 0$ otherwise. This functional equation is still an equation in two catalytic variables.

The differential classification

The problem is still to determine the classification of the generating function $Q(x,y)$, where it fits in the algebraic-differential hierarchy

$$\text{rational} \subset \text{algebraic} \subset \text{D-finite} \subset \text{D-algebraic}.$$

As seen in Chapter 1, setting the weights d_v, a and b to 1 (which amounts to ignoring the interaction and weights statistics, and is the original setting of the systematic classification), the classification of small steps models was completed in 2018. The methods of this first classification extend when considering arbitrary positive real weights d_v, and it is now complete as well.

Some of these techniques may in turn be adapted to the study of walks with the interacting boundaries statistics, where we allow other values of a and b, and the weighted models of walks with interacting boundaries that have been studied up to now rely on the finiteness of the classic group of the walk. This is the case in [TOR14], where the walks with interacting boundaries are completely solved for one specific model (the reversed Gessel model, also called Gouyou-Beauchamps) for all weights $a, b > 0$, establishing the full phase diagram. This is also the case in [BOX21], where the authors fully solve the Kreweras and reverse Kreweras walks with interaction for any value of the Boltzmann weights. Finally, in [BOR19], the authors systematically investigate the models having a finite group, for some Boltzmann weights, mainly (a, a), $(1, b)$, $(a, 1)$ and (a, b) for a and b algebraically independent over $\mathbb{Q}$, giving upper bounds on the complexity of the generating function $Q(1, 1)$. We propose to treat a case with an infinite group, and for nongeneric weights $d_{i,j}$, a and b.

6.1.2 Genus zero models

The authors of [DHRS21] define five models called the *genus zero models* (the terminology is explained in the next paragraph). They are listed in Figure 6.0.1 above, and they will be referenced in this chapter as $\mathcal{S}_1$, $\mathcal{S}_2$, etc. The goal of the current chapter is to establish the full classification of walks with interacting boundaries based on these sets of steps.

For these sets of steps, we may perform simplifications on the general functional equation (6.1.1). First, the Laurent polynomial is of the form

$$S(x, y) = d_{1,-1}\tfrac{x}{y} + d_{-1,1}\tfrac{y}{x} + d_{1,0}x + d_{0,1}y + d_{1,1}xy,$$

with $d_{1,-1}$ and $d_{-1,1}$ always nonzero, and at least one of the $d_{0,1}$, $d_{1,0}$ or $d_{1,1}$ nonzero. Moreover, we have that $A_{-1}(x) = d_{1,-1}x$ and $B_{-1}(y) = d_{-1,1}y$, and since no model contains the step $(-1, -1)$, the variable ε is always zero. Finally, for every model of Figure 6.0.1, the power series $Q(0, 0)$ is equal to 1. Indeed, any nontrivial walk one of these models has an ending point (i, j) satisfying $i + j > 0$ by an easy induction. Hence setting both x and y to 0 in $Q(x, y)$ leaves only the term in t^0, which is equal to 1. Summarizing these simplifications, the functional equation (6.1.1) can be rewritten as

$$\begin{aligned} K(x, y)Q(x, y) = {} & \frac{xy}{ab}(a + b - ab) \\ & + x\left(y - \frac{y}{a} - td_{1,-1}x\right) Q(x, 0) + y\left(x - \frac{x}{b} - td_{-1,1}y\right) Q(0, y). \end{aligned}$$

We now introduce the following notations:

$$\begin{aligned} A &\overset{\text{def}}{=} 1 - \frac{1}{a}, & B &\overset{\text{def}}{=} 1 - \frac{1}{b}, & \omega &\overset{\text{def}}{=} \frac{1}{ab}(a + b - ab), \\ \gamma_1(x, y) &\overset{\text{def}}{=} \frac{A}{x} - \frac{td_{1,-1}}{y}, & \gamma_2(x, y) &\overset{\text{def}}{=} \frac{B}{y} - \frac{td_{-1,1}}{x}, & \gamma(x, y) &\overset{\text{def}}{=} \frac{\gamma_1(x, y)}{\gamma_2(x, y)}. \end{aligned} \tag{6.1.2}$$

With these notations, the functional equation can finally be rewritten as

$$K(x, y)Q(x, y) = \omega xy + x^2y\gamma_1(x, y)Q(x, 0) + xy^2\gamma_2(x, y)Q(0, y). \tag{6.1.3}$$

In the remaining of the section, we show how to exploit further the particularity of the genus 0 models to study (6.1.3), and in the end classify $Q(x, y)$ for any weighting on one of these set of steps.

The kernel curve for models with genus 0

We are now going to take the geometric approach presented in Section 2.1, and use the kernel curve $\overline{E_t}$, that is the zero-locus of $\tilde{K}(x,y)$. Recall from Proposition 2.1.1 that there the kernel curve has either genus 0 or genus 1. To study the models whose curve has genus 0, it is enough to consider the five fundamental sets of steps of Figure 6.0.1. We thus recall the specifics on this curve when it has genus 0 (thus expanding the first point of Proposition 2.1.1).

When the kernel curve has genus 0, the authors of [DHRS21] construct a specific rational parametrization $\phi : \mathbb{P}^1 \rightarrow \overline{E_t}$. We summarize basic facts and vocabulary on this parametrization, following Section 4.1 of [DHRS21].

For the models of genus zero, the group of the walk acts of $\overline{E_t}$ through the parametrization $\phi : \mathbb{P}^1 \rightarrow \overline{E_t}$. Its generators ι_1 and ι_2 have the following expression in $\mathbb{P}^1 \times \mathbb{P}^1$.

$$\iota_1([1:x_1],[1:y_1]) = \left([1:x_1], \left[1 : \frac{d_{-1,1}x_1^2 + d_{0,1}x_1 + d_{1,1}}{d_{1,-1}y_1}\right]\right), \tag{6.1.4}$$

$$\iota_2([1:x_1],[1:y_1]) = \left(\left[1 : \frac{d_{1,-1}y_1^2 + d_{1,0}y_1 + d_{1,1}}{d_{-1,1}x_1}\right], [1:y_1]\right). \tag{6.1.5}$$

Note that we choose to write them for points of $\mathbb{P}^1 \times \mathbb{P}^1$ written in homogeneous coordinates $([1:x_1],[1:y_1])$ for reasons detailed in Section 6.4. These two involutions induce an automorphism σ of $\overline{E_t}$ defined as

$$\sigma \stackrel{\text{def}}{=} \iota_2 \circ \iota_1.$$

We are now going to summarize the properties of the parametrization ϕ of $\overline{E_t}$ defined in [DHRS21], and that we will use in the present chapter. It is constructed so that the action of the group of the walk lifts through ϕ in a nice way.

Proposition 6.1.1 (Section 4.1 of [DHRS21]). *For any model of Figure 6.0.1 and weighting d_v, and any real number $t \in]0,1[$ transcendental over $\mathbb{Q}(d_{i,j}, a, b)$, the following assertions hold.*

1. *There exists a rational parametrization $\phi : s \mapsto (x(s), y(s))$ of $\mathbb{P}^1$ onto $\overline{E_t}$. The fractions $x(s)$ and $y(s)$ both belong to $\overline{\mathbb{F}(t)}(s)$, where $\overline{\mathbb{F}(t)}$ is the algebraic closure of $\mathbb{F}(t)$.*

2. *The parametrization ϕ is injective everywhere except for $\phi(0) = \phi(\infty) = \Omega$, with $\Omega \stackrel{\text{def}}{=} ([0:1],[0:1])$, i.e. $(x(0), y(0)) = (x(\infty), y(\infty)) = \Omega$.*

3. *The divisors (Proposition 2.1.4) of the functions $x \stackrel{\text{def}}{=} x(s)$ and $y \stackrel{\text{def}}{=} y(s)$ on the curve $\mathbb{P}^1$ are*

$$(x) = 0 + \infty - Q_1 - Q_2, \qquad (y) = 0 + \infty - Q_3 - Q_4,$$

for some points $Q_i \neq 0, \infty$ of $\mathbb{P}^1$.

4. *The group lifts in the following way. There exists a real number $q \notin \{-1,1\}$, with q algebraic over $\mathbb{Q}(d_{i,j}, t)$, such that for all $s \in \mathbb{P}^1$ one has*

$$\iota_1(\phi(s)) = \phi(\tfrac{1}{s}) \text{ and } \iota_2(\phi(s)) = \phi(\tfrac{q}{s}).$$

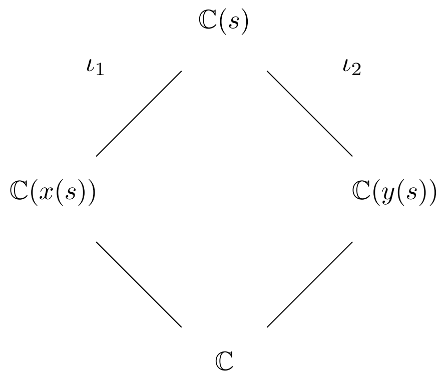
When the context is clear, we will also denote by ι_1, ι_2 and σ the automorphisms on $\mathbb{P}^1$ defined by

$$\iota_1(s) \stackrel{\text{def}}{=} \tfrac{1}{s} \qquad \iota_2(s) \stackrel{\text{def}}{=} \tfrac{q}{s} \qquad \sigma(s) \stackrel{\text{def}}{=} qs.$$

As the multiplicative order of q is infinite, so is the order of σ. Moreover, the only points of $\mathbb{P}^1$ whose orbit under the action of σ is finite are 0 and ∞.

Proposition 2.1.13 can be made more specific on the models of genus 0.

Proposition 6.1.2 (Section 1.3 of [DHRS21])**.** *The function field $\mathbb{C}(s)/\mathbb{C}$ of $\mathbb{P}^1$ has the following lattice.*



1. *The extension $\mathbb{C}(s)/\mathbb{C}(x(s))$ is Galois of degree 2, with Galois group generated by the involution ι_1. This means that if $h = h^{\iota_1}$, then $h \in \mathbb{C}(x(s))$.*
2. *Similarly, the extension $\mathbb{C}(s)/\mathbb{C}(y(s))$ is Galois of degree 2, with Galois group generated by the involution ι_2.*
3. *The field of constants $\mathbb{C}$ is the intersection $\mathbb{C}(x(s)) \cap \mathbb{C}(y(s))$, and as a result it is also the subfield of functions fixed by ι_1 and ι_2. Moreover, if $f(s)$ is fixed by σ, then $f(s)$ belongs to $\mathbb{C}$, σ having infinite order.*

6.1.3 The q-difference equations

We now fix the setting in which we can evaluate the functional equation (6.1.3) for $Q(x,y)$ on the curve $\overline{E_t}$ through the parametrization ϕ, in the manner of Section 2.1 of [DHRS21]. Once the conditions of this evaluation are fixed, this will transform the catalytic equation on $Q(x,y)$ (a power series in x and y) into a functional equation relating $Q(x(s),0)$ and $Q(0,y(s))$ (meromorphic functions over $\mathbb{C} \subset \mathbb{P}^1$). The symmetries of $x(s)$ and $y(s)$ with respect to the group (Proposition 6.1.1) will then allow us to construct a functional equation on $Q(x(s), 0)$ only.

We first study the divisors of the functions

$$\widetilde{\gamma}_1(s) \stackrel{\text{def}}{=} \gamma_1(x(s), y(s)) \quad \widetilde{\gamma}_2(s) \stackrel{\text{def}}{=} \gamma_2(x(s), y(s)) \tag{6.1.6}$$

on $\mathbb{P}^1$ through a routine computation. Recall that

$$\gamma_1(x,y) = \frac{A}{x} - t\frac{d_{1,-1}}{y} \qquad\qquad \gamma_2(x,y) = \frac{B}{y} - t\frac{d_{-1,1}}{x},$$

as defined in (6.1.2).

Proposition 6.1.3. *For any real number t defined as in Proposition 6.1.1, the extensions $\mathbb{C}(s)/\mathbb{C}(\widetilde{\gamma}_1)$ and $\mathbb{C}(s)/\mathbb{C}(\widetilde{\gamma}_2)$ have degree two, the functions $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ having the following divisors*

$$(\widetilde{\gamma}_1) = P_1 + P_2 - 0 - \infty, \qquad\qquad (\widetilde{\gamma}_2) = P_3 + P_4 - 0 - \infty.$$

The points $P_1, \dots, P_4$ are distinct from 0 and ∞.

Proof. In this proof, we write $x = x(s)$ and $y = y(s)$, these functions thus satisfying $K(x, y) = 0$ for $K(X, Y)$ the kernel polynomial. We only perform the proof for $\widetilde{\gamma}_1 = \frac{A}{x} - \frac{td_{1,-1}}{y}$, the study of $\widetilde{\gamma}_2$ being symmetric. The computations belong to the joint Maple worksheet. We are going to prove that the minimal polynomial of x over $\mathbb{C}(\widetilde{\gamma}_1)$ has degree 2, thus showing that $[\mathbb{C}(x) : \mathbb{C}(\widetilde{\gamma}_1)] = 2$. We will then prove that $\widetilde{\gamma}_1$ has the announced poles 0 and ∞.

We first produce a vanishing polynomial of x over $\mathbb{C}(\widetilde{\gamma}_1)$. By definition, the polynomial $K(X, y) = 0$ is a vanishing polynomial of x over $\mathbb{C}(y)$. Hence, expressing y in terms of x and $\widetilde{\gamma}_1$ (which we can do since $td_{1,-1}$ is nonzero), we are left to consider the polynomial

$$\begin{aligned} P(X) = {} & (d_{1,1}d_{1,-1}t^2 - d_{1,0}t\widetilde{\gamma}_1 + \widetilde{\gamma}_1^2)X^2 \\ & + (d_{0,1}d_{1,-1}t^2 + Ad_{1,0}t + (1-2A)\widetilde{\gamma}_1)X \\ & + d_{1,-1}d_{-1,1}t^2 + A^2 - A, \end{aligned}$$

which by construction is a vanishing polynomial of x with coefficients in $\mathbb{C}[\widetilde{\gamma}_1] \subset \mathbb{C}(\widetilde{\gamma}_1)$. Moreover, $P(X)$ is nonzero, for its constant coefficient $d_{1,-1}d_{-1,1}t^2 + A^2 - A \in \mathbb{F}[t]$ is nonzero. Indeed, the real number t is transcendental over $\mathbb{F}$ and the coefficient $d_{1,-1}d_{-1,1}$ of t is nonzero for all genus zero models (Figure 6.0.1). Hence, x is algebraic over $\mathbb{C}(\widetilde{\gamma}_1)$.

We now show that $P(X)$ is irreducible in $\mathbb{C}(\widetilde{\gamma}_1)[X]$. First note that the leading coefficient $d_{1,1}d_{1,-1}t^2 - d_{1,0}t\widetilde{\gamma}_1 + \widetilde{\gamma}_1^2$ of $P(X)$ is nonzero. Indeed, it is a nonzero polynomial of $\mathbb{C}[\widetilde{\gamma}_1]$, with $\widetilde{\gamma}_1$ transcendental over $\mathbb{C}$ (the function x is both transcendental over $\mathbb{C}$ and algebraic over $\mathbb{C}(\widetilde{\gamma}_1)$). Hence, $P(X)$ is a degree two polynomial. In order to show it is irreducible, we thus compute its discriminant $\Delta \in \mathbb{C}(\widetilde{\gamma}_1)$ and show that it cannot be a square in $\mathbb{C}(\widetilde{\gamma}_1)$. The discriminant is expressed as follows:

$$\begin{aligned}\Delta = &(-4d_{1,-1}d_{-1,1}t^2+1)\widetilde{\gamma}_1^2\\ &+(4d_{1,0}d_{1,-1}d_{-1,1}t^3-4Ad_{0,1}d_{1,-1}t^2+2d_{0,1}d_{1,-1}t^2-2Ad_{1,0}t)\widetilde{\gamma}_1\\ &+(d_{0,1}^2d_{1,-1}^2t^4-4d_{1,1}d_{1,-1}^2d_{-1,1}t^4+2Ad_{0,1}d_{1,0}d_{1,-1}t^3\\ &\quad+A^2d_{1,0}^2t^2-4A^2d_{1,1}d_{1,-1}t^2+4Ad_{1,1}d_{1,-1}t^2).\end{aligned}$$

We see that Δ belongs to $\mathbb{C}[\widetilde{\gamma}_1]$, hence Δ is a square in $\mathbb{C}(\widetilde{\gamma}_1)$ if and only if Δ is a square in $\mathbb{C}[\widetilde{\gamma}_1]$. In turn, as Δ has degree two as a polynomial in $\widetilde{\gamma}_1$ (the leading coefficient $-4d_{1,-1}d_{-1,1}t^2+1$ is nonzero since t is transcendental over $\mathbb{F}$), Δ is a square if and only if its discriminant δ with respect to the variable $\widetilde{\gamma}_1$ is zero. The discriminant δ factors in $\mathbb{Q}[d_{i,j},A][t]$ into $\delta = 16t^2f_1f_2f_3$ (the factors f_i written in the table below).

	Polynomial
f_1	$d_{1,-1}$
f_2	$d_{1,-1}d_{-1,1}t^2+A(A-1)$
f_3	$(d_{0,1}^2d_{1,-1}+d_{1,0}^2d_{-1,1}-4d_{1,1}d_{1,-1}d_{-1,1})t^2+d_{0,1}d_{1,0}t+d_{1,1}$

As t is transcendental over $\mathbb{Q}(d_{i,j},a,b)$, δ is zero if and only if one f_i is zero if and only if all coefficients of one f_i viewed as a polynomial in $\mathbb{F}[t]$ are zero. Define an ideal I of $\mathbb{Z}[d_{i,j},A]$ by $I = d_{1,-1}d_{-1,1}(d_{1,1},d_{0,1},d_{1,0})$, and for $i \in \{1,2,3\}$ the ideal J_i of $\mathbb{Z}[d_{i,j},A]$ generated by the coefficients in t of f_i. Then one sees that $I \subset J_1$, $I \subset J_2$, and finally through elimination that $I^3 \subset J_3$. Hence, if δ is zero, then $(d_{1,-1},d_{-1,1},d_{0,1},d_{1,0},d_{1,1},A)$ must satisfy either $d_{1,-1}=0$, or $d_{-1,1}=0$, or $d_{1,0}=d_{0,1}=d_{1,1}=0$, which is never the case given the constraints on the supports (Figure 6.0.1). Therefore, δ is always nonzero, Δ is never a square, and we conclude that $P(X)$ is always irreducible in $\mathbb{C}(\widetilde{\gamma}_1)[X]$. Thus, since P is an irreducible vanishing polynomial of x, and since $\mathbb{C}(x,\widetilde{\gamma}_1)=\mathbb{C}(x,y)$, this proves that $[\mathbb{C}(x,y):\mathbb{C}(\widetilde{\gamma}_1)] = [\mathbb{C}(x,\widetilde{\gamma}_1):\mathbb{C}(\widetilde{\gamma}_1)] = 2$.

We now conclude on the poles of $\widetilde{\gamma}_1$. By Proposition 6.1.1, the divisors of x and y are respectively $(x) = 0+\infty-Q_1-Q_2$ and $(y)=0+\infty-Q_3-Q_4$ for some points Q_i. Hence, the poles of $\widetilde{\gamma}_1 = \frac{A}{x} - t\frac{d_{1,-1}}{y}$ are either 0 or ∞, which both have order at most 1. But by Proposition 2.1.4, $\deg(\widetilde{\gamma}_1)_\infty = [\mathbb{C}(s):\mathbb{C}(\widetilde{\gamma}_1)] = 2$, thus we conclude that $\widetilde{\gamma}_1$ has these two poles, and thus $(\widetilde{\gamma}_1)_\infty = 0+\infty$. ☐

We now determine for which real numbers t the composition of the functional equation (6.1.3) with $(x(s),y(s))$ (which both depend on t) is well defined.

Proposition 6.1.4. *There exists a positive real number $r>0$ such that for every real number $t<r$ transcendental over $\mathbb{F}$, there exist two open sets U_0 and U_∞ of $\mathbb{P}^1$ such that $0\in U_0$, $\infty \in U_\infty$, and so that the functions $Q(x(s),y(s))$, $Q(x(s),0)$, $Q(0,y(s))$, are analytic on $U_0\cup U_\infty$. Moreover, there exists an open set V satisfying $0\in V\subset U_0$ such that $\iota_2(V)\subset U_\infty$ and $\sigma^{-1}(V) \subset U_0$.*

Proof. We are going to show that for t small enough, the series $Q(x,y) \in \mathbb{C}[x,y][\![t]\!]$ is convergent on $\{(x,y) \in \mathbb{C} : |x|,|y| < 1\}$. Let $|x|,|y| < 1$, and $M \stackrel{\text{def}}{=} \sup\{|d_{i,j}|,|a|,|b|\}$ (note that M is positive). Then each walk w of length n contributes to a value of norm at most M^{2n} to the coefficient of t^n in $Q(x,y)$. Indeed, each performed step involves at most two weights (one $d_{i,j}$ and possibly one additional a or b). As the steps are small, the coefficient corresponding to w has norm at most $M^{2n}|x|^n|y|^n \leq M^{2n}$, for $|x|,|y| < 1$. Moreover, the set of steps of any of the considered weighted models is finite of cardinal at most 5 (Figure 6.0.1), hence the coefficient of t^n in $Q(x,y)$ has norm of at most $(5M^2)^n$. Hence, when $|x|,|y| < 1$, the power series $Q(x,y) \in \mathbb{C}(x,y)[\![t]\!]$ has a positive radius of convergence $\rho \geq \frac{1}{5M^2}$. We thus fix a real number t so that $0 < t < \rho$ and t is transcendental over the field $\mathbb{F}$. Since the coefficients in t of $Q(x,y)$ are polynomials in x and y, the function $Q(x,y)$ is analytic in x and y at $(0,0)$.

We now study the convergence of the composition of the functions appearing in (6.1.3) with the parametrization $\phi(s) = (x(s),y(s))$. First, the functions $x(s)$ and $y(s)$ belong to $\mathbb{C}(s)$ with $x(0) = y(0) = x(\infty) = y(\infty) = 0$, thus they are both analytic at the points 0 and ∞. Thus, by composition, the functions $Q(x(s),y(s))$, $Q(x(s),0)$ and $Q(0,y(s))$ are analytic at 0 and ∞. This proves the existence of the two announced open sets $U_0 \ni 0$ and $U_\infty \ni \infty$.

Finally, we construct V to be $U_0 \cap \iota_2^{-1}(U_\infty) \cap \sigma(U_0)$. This is open because $\iota_2(s) = \frac{q}{s}$ and $\sigma(s) = qs$, which are continuous functions in $\mathbb{P}^1 \mapsto \mathbb{P}^1$. Moreover, V contains 0 because $\iota_2(\infty) = \iota_1(\infty) = \sigma(0) = \sigma^{-1}(0) = 0$. □

We now fix some small enough t transcendental over $\mathbb{F}$ prescribed by Proposition 6.1.4, and a parametrization ϕ accordingly. The evaluation of (6.1.3) on $(x(s),y(s))$ for s in $U_0 \cup U_\infty$ is thus well defined, yielding after dividing by $x(s)y(s)$ the following equation of meromorphic functions on $U_0 \cup U_\infty$:

$$0 = \omega + x(s)\widetilde{\gamma}_1(s)Q(x(s),0) + y(s)\widetilde{\gamma}_2(s)Q(0,y(s)). \tag{6.1.7}$$

Define two analytic functions on the open set V by

$$\breve{F}(s) \stackrel{\text{def}}{=} x(s)Q(x(s),0), \qquad \breve{G}(s) \stackrel{\text{def}}{=} y(s)Q(0,y(s)).$$

We now use (6.1.7) to construct meromorphic continuations $\widetilde{F}$ of $\breve{F}$ and $\widetilde{G}$ of $\breve{G}$ to the whole complex plane $\mathbb{C}$, by showing like in [DHRS20] that $\breve{F}$ satisfies a q-difference equation. We introduce the function

$$\widetilde{\gamma}(s) \stackrel{\text{def}}{=} \gamma(x(s),y(s)) = \frac{\widetilde{\gamma}_1(s)}{\widetilde{\gamma}_2(s)}, \tag{6.1.8}$$

where $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ were introduced in (6.1.6), and we rewrite (6.1.7) as follows for s in V.

$$-y(s)Q(0,y(s)) = \frac{\omega}{\widetilde{\gamma}_2(s)} + \widetilde{\gamma}(s)x(s)Q(x(s),0). \tag{6.1.9}$$

By Proposition 6.1.4, if s is in V then $\frac{q}{s} = \iota_2(s) \in U_\infty$, hence from (6.1.7) we also have the following equation for s in V:

$$-y(\tfrac{q}{s})Q(0, y(\tfrac{q}{s})) = \frac{\omega}{\widetilde{\gamma}_2(\frac{q}{s})} + \widetilde{\gamma}(\tfrac{q}{s})x(\tfrac{q}{s})Q(x\left(\tfrac{q}{s}\right), 0). \tag{6.1.10}$$

We now use the symmetries of the functions $x(s)$ and $y(s)$. For all s in V, we have by Proposition 6.1.1 that $y(\frac{q}{s}) = y(s)$, hence $-y(\frac{q}{s})Q(0, y(\frac{q}{s})) = -y(s)Q(0, y(s))$. Moreover, we also have $x(\frac{q}{s}) = x(\frac{s}{q})$, hence $x(\frac{q}{s})Q(x(\frac{q}{s}), 0) = x(\frac{s}{q})Q(x(\frac{s}{q}), 0)$. Finally, as s is in V, the complex number $\frac{s}{q} = \sigma^{-1}(s)$ is also in U_0 by Proposition 6.1.4, so we can replace $x(\frac{q}{s})Q(x(\frac{q}{s}), 0)$ with $\breve{F}(\frac{s}{q})$ in (6.1.10). Hence, eliminating $y(s)Q(0, y(s))$ between (6.1.9) and (6.1.10) yields the following q-difference equation on $\breve{F}(s)$ for all s in V:

$$\breve{F}(\tfrac{s}{q}) = \frac{\widetilde{\gamma}}{\widetilde{\gamma}^{\iota_2}}(s)\breve{F}(s) + \left(\frac{\omega}{\widetilde{\gamma}_2(s)} - \frac{\omega}{\widetilde{\gamma}_2^{\iota_2}(s)}\right)\frac{1}{\widetilde{\gamma}^{\iota_2}(s)}.$$

As the absolute value of q is not equal to 1 (Proposition 6.1.1), this functional equation allows us to construct a unique continuation $\widetilde{F}$ of $\breve{F}$ meromorphic on the whole complex plane $\mathbb{C}$, satisfying the same equation:

$$\widetilde{F}(\tfrac{s}{q}) = \frac{\widetilde{\gamma}}{\widetilde{\gamma}^{\iota_2}}(s)\widetilde{F}(s) + \left(\frac{\omega}{\widetilde{\gamma}_2(s)} - \frac{\omega}{\widetilde{\gamma}_2^{\iota_2}(s)}\right)\frac{1}{\widetilde{\gamma}^{\iota_2}(s)}. \tag{6.1.11}$$

Indeed, assuming that $\widetilde{F}$ is a meromorphic continuation of $\breve{F}$ on some open set U, the functional equation (6.1.11) relates $\widetilde{F}(s)$ with $\widetilde{F}(\frac{s}{q})$ over $\mathbb{C}(s)$, which allows us to extend uniquely $\widetilde{F}$ as a meromorphic function on $qU \cup U \cup q^{-1}U$. As $|q| \neq 1$ (Proposition 6.1.1), we have that $\bigcup_{n\in\mathbb{Z}} q^n U = \mathbb{C}$, hence this process gives a unique meromorphic continuation $\widetilde{F}$ of $\breve{F}$ on $\mathbb{C}$.

Now, the functions $\breve{F}$ and $\breve{G}$ satisfy the linear relation (6.1.7) over $\mathbb{C}(s)$. This relation provides a unique meromorphic continuation $\widetilde{G}$ of $\breve{G}$ to $\mathbb{C}$ such that $\widetilde{F}$ and $\widetilde{G}$ satisfy

$$\widetilde{\gamma}_1(s)\widetilde{F}(s) + \widetilde{\gamma}_2(s)\widetilde{G}(s) + \omega = 0. \tag{6.1.12}$$

Finally, from (6.1.11) and (6.1.12), it is easy to see that the function $\widetilde{G}$ satisfies the following q-difference equation:

$$\widetilde{G}(qs) = \frac{\widetilde{\gamma}^{\iota_1}}{\widetilde{\gamma}}(s)\widetilde{G}(s) + \left(\frac{\omega}{\widetilde{\gamma}_1(s)} - \frac{\omega}{\widetilde{\gamma}_1^{\iota_1}(s)}\right)\widetilde{\gamma}^{\iota_1}(s). \tag{6.1.13}$$

6.1.4 The D-algebraicity of $Q(x, y)$, $\widetilde{F}(s)$ and $\widetilde{G}(s)$

The algebraic-differential properties of the formal power series $Q(x, y)$, $Q(x, 0)$ and $Q(0, y)$ and their meromorphic counterparts $\widetilde{F}(s)$ and $\widetilde{G}(s)$ are related. The following proposition relates the x and y-D-algebraicity of $Q(x, y)$ with the s-D-algebraicity of $\widetilde{F}(s)$ and $\widetilde{G}(s)$. As t is a fixed real number, the study of the t-D-algebraicity of $Q(x, y, t)$ is not easily related to the properties of $\widetilde{F}(s)$ and $\widetilde{G}(s)$, and rigid parametrizations are needed (see [DH21]), which are not implemented in this chapter.

Proposition 6.1.5. *For $t > 0$ as in Proposition 6.1.4, and a weighting $(d_v)_v$, a and b, the following statements are equivalent:*

(a) $Q(x,0)$ *is x-D-algebraic,*

(a′) $Q(0,y)$ *is y-D-algebraic,*

(b) $\widetilde{F}(s)$ *is s-D-algebraic,*

(b′) $\widetilde{G}(s)$ *is s-D-algebraic,*

(c) $Q(x,y)$ *is x-D-algebraic for all y,*

(c′) $Q(x,y)$ *is y-D-algebraic for all x.*

Proof. We recall that on the open set V, we have

$$\widetilde{F}(s) = x(s)Q(x(s),0), \qquad \widetilde{G}(s) = y(s)Q(0,y(s)).$$

Up to a restriction of V, the maps $x(s)$ and $y(s)$ are biholomorphisms on V. The equivalence between (a) and (b) (resp. (a') and (b')) now follows from Lemmas 6.3 and 6.4 of [DHRS18].

Moreover, the functions $\widetilde{F}$ and $\widetilde{G}$ are linearly related over $\mathbb{C}(s)$ by Equation (6.1.12), which shows the equivalence between (b) and (b'), and thus the equivalence between (a), (a'), (b) and (b').

Finally, (c) is equivalent to (a) from Equation (6.1.3). Indeed, $\partial_x Q(0,y) = 0$, so $Q(0,y)$ is x-D-algebraic. Hence, since $x^2 y\gamma_1(x,y)$ and $K(x,y)$ are both nonzero elements of $\mathbb{C}(x,y)$, the closure properties of the x-D-algebraic class imply that $Q(x,y)$ is x-D-algebraic if and only if $Q(x,0)$ is. Similarly, (c') is equivalent to (a'). □

Therefore, determining the differential nature of the function $Q(x,y)$ is equivalent to determining the differential nature of either $\widetilde{F}(s)$ or $\widetilde{G}(s)$, which satisfy functional equations with more structure: q-difference equations. Equations (6.1.11) and (6.1.13) do not completely characterize $\widetilde{F}(s)$ and $\widetilde{G}(s)$, so we will use in a crucial way the fact that they continue the functions $x(s)Q(x(s),0)$ and $y(s)Q(0,y(s))$, on which we have some grasp (information on poles near zero mainly).

Finally, we will often make use of the following proposition, which allows us to go from meromorphic functions on $\mathbb{C}$ to power series, mostly to obtain equations on $Q(x,0)$ (resp. $Q(0,y)$) from equations on $\widetilde{F}(s)$ (resp. $\widetilde{G}(s)$).

Proposition 6.1.6. *Assume that a Laurent series $H(x) \in \mathbb{C}((x))$ induces a meromorphic function at $x = 0$. If $H(x(s)) = 0$ or if $H(y(s)) = 0$ for s near 0, then $H(x) = 0$.*

Proof. Let W be a neighborhood of 0 such that $H(x(s)) = 0$ for all s in $W \setminus \{0\}$. The function $x : \mathbb{C} \to \mathbb{C}$ is non-constant and analytic at 0 with $x(0) = 0$. Hence by the open mapping theorem for holomorphic functions, the image of W under x is an open neighborhood of 0. Thus, the analytic function $H(x)$ is locally zero at 0, hence zero by analytic continuation. The argument is similar for $H(y(s)) = 0$. □

6.2 Classification strategy

In the previous section, we have constructed two q-difference equations (6.1.11) and (6.1.13) satisfied by two meromorphic functions on $\mathbb{C}$, whose differential properties reflect those of the generating functions of quadrant walks $Q(x,y)$. There are now many results on the differential transcendence of power series solution to a q-difference equation. One of the first of those results was proved by Ishizaki for the solutions of equations of the form $y(qs) = a(s)y(s) + b(s)$ [Ish98]. We apply it to equations (6.1.11) and (6.1.13):

Proposition 6.2.1. *The following statements are equivalent:*

1. *$\widetilde{F}$ and $\widetilde{G}$ are D-algebraic over $\mathbb{C}(s)$.*
2. *$\widetilde{F}$ and $\widetilde{G}$ are in $\mathbb{C}(s)$.*

Proof. The real number q is not a root of unit, the coefficients of (6.1.11) and (6.1.13) belong to $\mathbb{C}(s)$, and the functions $\widetilde{F}$ and $\widetilde{G}$ are meromorphic at $s = 0$. Hence we may apply Theorem 1.2 of [Ish98] to the two equations, which shows the claim. □

Thus, to investigate the D-algebraicity of $\widetilde{F}$ (or equivalently $\widetilde{G}$), the strategy that we are going to explain in this section will consist in determining for which weighted models these functions can be rational. In the earlier paper applying this strategy [DHRS20] (corresponding to the case without the interaction statistics $a = b = 1$), the authors find no rational solution. Whether it exists is highly dependent on the coefficients of the q-difference equations considered. For the classification of the models with the five supports of Figure 6.0.1, we find a general strategy which allows us to handle almost all the cases uniformly, its culmination being the classification in Theorem 6.5.8.

Because of the symmetries of the coefficients of equations (6.1.11) and (6.1.13), we will reduce the study of rational solutions to these equations to what we call *decoupling equations*. Recall as in 2.1.1 that the curve $\overline{E_t} \subset \mathbb{P}^1 \times \mathbb{P}^1$ has a uniformization $\phi : \mathcal{C} \to \mathbb{P}^1$ by a smooth curve $\mathcal{C}$. Consider some fraction $h(x,y) \in \mathbb{C}(x,y)$. The problem is to find two fractions $f(x)$ and $g(y)$ so that the following equation holds for all points P in $\mathcal{C}$,

$$h(x(P), y(P)) = f(x(P)) + g(y(P)).$$

This is called an *additive decoupling of h*, this notion being introduced in [BBR21]. Likewise, one can wonder if there exist $f(x)$ and $g(y)$ so that for all points P in $\mathcal{C}$,

$$h(x(P), y(P)) = f(x(P))g(y(P)).$$

This is called a *multiplicative decoupling of h* (introduced in [BEFHR25]).

The existence of such decouplings for a fraction $h(x,y)$ plays an important role in the classification of generating functions enumerating walks. For instance, in the case of quadrant walks with small steps without interacting boundaries, algebraicity is characterized by the finiteness of the group, and the fact that the fraction xy admits an additive decoupling (see [DER24]). In our case, the equations that appear are generalizations of the decoupling equations above, mixing the additive and multiplicative form, and they serve the same purpose: we will see that the fact that they admit a solution or not determines the position of $Q(x,y)$ in the differential hierarchy. This explains our choice of terminology, as we introduce them now.

Lemma 6.2.2. *Assume that the functions $\widetilde{F}(s)$ and $\widetilde{G}(s)$ (defined in Section 6.1.3) are rational. In this case, define the following elements of $\mathbb{C}(s)$:*

$$\widetilde{f}(s) \overset{\text{def}}{=} \frac{1}{2}\left(\widetilde{F}(s) + \widetilde{F}^{\iota_1}(s)\right), \qquad \widetilde{g}(s) \overset{\text{def}}{=} \frac{1}{2}\left(\widetilde{G}(s) + \widetilde{G}^{\iota_2}(s)\right),$$

$$\widetilde{f}_h(s) \overset{\text{def}}{=} \frac{1}{2}\left(\widetilde{F}(s) - \widetilde{F}^{\iota_1}(s)\right), \qquad \widetilde{g}_h(s) \overset{\text{def}}{=} \frac{1}{2}\left(\widetilde{G}(s) - \widetilde{G}^{\iota_2}(s)\right).$$

1. *The pair $(h_1(s), h_2(s)) = (\widetilde{f}(s), \widetilde{g}(s))$ satisfies the inhomogeneous equation*

$$\widetilde{\gamma}_1(s)h_1(s) + \widetilde{\gamma}_2(s)h_2(s) + \omega = 0 \text{ with } h_1^{\iota_1} = h_1 \text{ and } h_2^{\iota_2} = h_2. \qquad (E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega})$$

2. *The pair $(h_1(s), h_2(s)) = (\widetilde{f}_h(s), \widetilde{g}_h(s))$ satisfies the homogeneous equation*

$$\widetilde{\gamma}_1(s)h_1(s) + \widetilde{\gamma}_2(s)h_2(s) = 0 \text{ with } h_1^{\iota_1} = -h_1 \text{ and } h_2^{\iota_2} = -h_2. \qquad (E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2})$$

We refer to equations ($E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega}$) and ($E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2}$) as the decoupling equations.

Proof. Assuming that $\widetilde{F}$ and $\widetilde{G}$ belong to $\mathbb{C}(s)$, we first find a relation between $\widetilde{F}^{\iota_1}$ and $\widetilde{G}^{\iota_2}$ (composition with ι_1 and ι_2 is always well defined for rational functions). Recall that $\widetilde{F}(\frac{s}{q}) = \widetilde{F}^{\iota_1\iota_2}$, hence we may rewrite the q-difference equation (6.1.11) into

$$\left(\widetilde{\gamma}(s)\widetilde{F}^{\iota_1}(s) + \frac{\omega}{\widetilde{\gamma}_2(s)}\right)^{\iota_2} = \widetilde{\gamma}(s)\widetilde{F}(s) + \frac{\omega}{\widetilde{\gamma}_2(s)},$$

hence by applying ι_2 on both sides we obtain

$$\widetilde{\gamma}(s)\widetilde{F}^{\iota_1}(s) + \frac{\omega}{\widetilde{\gamma}_2(s)} = \left(\widetilde{\gamma}(s)\widetilde{F}(s) + \frac{\omega}{\widetilde{\gamma}_2(s)}\right)^{\iota_2}. \qquad (6.2.1)$$

Moreover, the linear relation (6.1.12) between $\widetilde{F}$ and $\widetilde{G}$ can be rewritten as

$$-\widetilde{G}(s) = \widetilde{\gamma}(s)\widetilde{F}(s) + \frac{\omega}{\widetilde{\gamma}_2(s)},$$

so by applying ι_2 we obtain

$$-\widetilde{G}^{\iota_2}(s) = \left(\widetilde{\gamma}(s)\widetilde{F}(s) + \frac{\omega}{\widetilde{\gamma}_2(s)}\right)^{\iota_2}. \qquad (6.2.2)$$

Eliminating the right-hand sides between (6.2.1) and (6.2.2), we extract the following relation between $\widetilde{F}^{\iota_1}$ and $\widetilde{G}^{\iota_2}$:

$$\widetilde{\gamma}_1(s)\widetilde{F}^{\iota_1}(s) + \widetilde{\gamma}_2(s)\widetilde{G}^{\iota_2}(s) + \omega = 0. \qquad (6.2.3)$$

We copy for convenience Equation (6.1.12):

$$\widetilde{\gamma}_1(s)\widetilde{F}(s) + \widetilde{\gamma}_2(s)\widetilde{G}(s) + \omega = 0. \tag{6.2.4}$$

Taking the mean of equations (6.2.3) and (6.2.4), one obtains ($E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega}$). Taking half the difference of equations (6.2.3) and (6.2.4), one obtains ($E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2}$). Indeed, ι_1 is an involution, hence $(\widetilde{F}^{\iota_1})^{\iota_1} = \widetilde{F}$, so $\widetilde{f}^{\iota_1} = \widetilde{f}$ and $\widetilde{f}_h^{\iota_1} = -\widetilde{f}_h$. The same argument applies to $\widetilde{g}$ and $\widetilde{g}_h$. □

Remark 6.2.3. From Proposition 6.1.2, the condition $\widetilde{f}^{\iota_1} = \widetilde{f}$ is equivalent to the condition that there exists $f(x) \in \mathbb{C}(x)$ such that $\widetilde{f}(s) = f(x(s))$. Likewise, the condition $\widetilde{f}^{\iota_1} = -\widetilde{f}$ asserts that there exists $\widetilde{f}(s)^2 = f(x(s))$ for some f, but that $\widetilde{f}(s)$ itself is not a function of $x(s)$. This explains the qualification of *decoupling equations*: they relate functions in two different variables. ∎

We will see in the remaining of the section how the study of the rational solutions of the decoupling equations gives information on the series $Q(x,y)$, either for showing its non-D-algebraicity in x and y, or for obtaining an explicit algebraic expression.

6.2.1 Showing non D-algebraicity

Our argument for showing non-D-algebraicity will rely on the fact that we know that the solution $Q(x,y)$ is a generating function of walks in the quadrant, and thus we may control the expansion of $Q(x,0)$ and $Q(0,y)$ around 0.

Lemma 6.2.4. *Let h be a fraction of $\mathbb{C}(s)$, and assume that the poles of h belong to $\{0,\infty\}$. If $h^{\iota_1} = h$, then there exists a Laurent polynomial $H(x) \in \mathbb{C}[1/x]$ such that $H(x(s)) = h(s)$. Analogously, if $h^{\iota_2} = h$, then there exists a Laurent polynomial $H(y) \in \mathbb{C}[1/y]$ such that $H(y(s)) = h(s)$.*

Proof. Let $h \in \mathbb{C}(s)$ be a function whose poles belong to $\{0,\infty\}$, such that $h^{\iota_1} = h$. The extension $\mathbb{C}(s)/\mathbb{C}(x(s))$ being Galois of Galois group generated by ι_1 by Proposition 6.1.1, there exists a fraction $H(x) \in \mathbb{C}(x)$ such that $H(x(s)) = h(s)$. Write $H(x) = \frac{U(x)}{V(x)}$ with U and V relatively prime polynomials, V monic, and let $u(s) \stackrel{\text{def}}{=} U(x(s))$, $v(s) \stackrel{\text{def}}{=} V(x(s))$. We write the polar divisor $(h)_\infty$ of h in two different ways.

First, we have by the assumption that $(h)_\infty = p \cdot 0 + q \cdot \infty$ for some nonnegative integers p and q. As $\iota_1(0) = \infty$ and $h^{\iota_1} = h$, we conclude that $p = q$, so by Proposition 6.1.1,

$$(h)_\infty = p \cdot 0 + p \cdot \infty = p \cdot (x(s))_0. \tag{6.2.5}$$

Moreover, as u and v are polynomials in $x(s)$, we have that $(u)_\infty = \deg_x U(x) \cdot (x(s))_\infty$. and $(v)_\infty = \deg_x V(x) \cdot (x(s))_\infty$. We also have that since the polynomials $U(x)$ and $V(x)$ are relatively prime, Bézout theorem implies the existence of some relation

$$U'(x)U(x) + V'(x)V(x) = 1$$

for $U'(x), V'(x) \in \mathbb{C}[x]$. Thus, by composing the relation with $x(s)$, we see that

$$u'(s)u(s) + v'(s)v(s) = 1.$$

For s_0 a pole of $x(s)$, $U(x(s_0))$ and $V(x(s_0))$ are both nonzero since $U(x)$ and $V(x)$ are polynomials. For s_0 not a pole of $x(s)$, then s_0 is not a pole of u', v', u, v, and we see that it cannot be a zero of both $u(s)$ and $v(s)$. Thus, since

$$\begin{aligned}(h) &= (u)_0 + (v)_\infty - (u)_\infty - (v)_0 \\ &= (u)_0 - (v)_0 + (\deg_x V(x) - \deg_x U(x)) \cdot (x(s))_\infty\end{aligned}$$

and the zeros of u and v don't compensate, we deduce that

$$(h)_\infty = (v)_0 - \min(\deg_x V(x) - \deg_x U(x), 0) \cdot (x(s))_\infty. \tag{6.2.6}$$

Therefore, equating (6.2.5) and (6.2.6), we obtain the conditions

(1) $\deg_x V(x) - \deg_x U(x) \geq 0$ (2) $(v)_0 = p \cdot (x(s))_0$.

Consider $w(s) \overset{\text{def}}{=} \frac{v(s)}{x(s)^p}$. Using (2), we compute its divisor as

$$\begin{aligned}(w) &= (v)_0 - (v)_\infty - p \cdot (x(s))_0 + p \cdot (x(s))_\infty \\ &= (p - \deg_x V(x)) \cdot (x(s))_\infty.\end{aligned}$$

Since $\deg(w) = 0$ (Proposition 2.1.4), we deduce that $2 \cdot (p - \deg_x V(x)) = 0$, and thus $(w) = 0$, which implies that w is a constant (Proposition 2.1.4). As $V(x)$ is monic, this implies that $V(x) = x^p$. Moreover condition (1) implies that $\deg_x U(x) \leq \deg_x V(x) = p$. We thus conclude that $H(x) = \frac{U(x)}{V(x)}$ belongs to $\mathbb{C}[1/x]$. The proof for $h^{\iota_2} = h$ is similar. □

Lemma 6.2.5. *Assume that the following two conditions hold:*

(1) For any pair of solutions (h_1, h_2) of $(E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega})$, then either the poles of h_1 are in $\{0, \infty\}$ or the poles of h_2 are in $\{0, \infty\}$.

(2) The only solution of $(E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2})$ is $(0, 0)$.

Then $Q(x, y)$ is non x-D-algebraic nor y-D-algebraic.

Proof. Assume that $Q(x, y)$ is x-D-algebraic or y-D-algebraic, then by Proposition 6.1.5 the functions $\widetilde{F}$ and $\widetilde{G}$ are rational. Hence, by Lemma 6.2.2, the pair $(\widetilde{f}, \widetilde{g})$ satisfies $(E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega})$ and the pair $(\widetilde{f}_h, \widetilde{g}_h)$ satisfies $(E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2})$, with $\widetilde{F} = \widetilde{f} + \widetilde{f}_h$ and $\widetilde{G} = \widetilde{g} + \widetilde{g}_h$.

From (1), assume without loss of generality that the poles of $\widetilde{f}$ are in the set $\{0, \infty\}$. Then by Lemma 6.2.4 applied to $\widetilde{f}$, there exists a Laurent polynomial $f(x) \in \mathbb{C}[1/x]$ such that $f(x(s)) = \widetilde{f}(s)$. Denote by $-d$ the valuation of $f(x)$, so that $P(x) \overset{\text{def}}{=} x^d f(x)$

is a polynomial of degree at most d. From (2), we have that $\widetilde{f}_h = 0$, hence $\widetilde{F}(s) = \widetilde{f}(s)$. Therefore, as $\widetilde{F}(s)$ is a continuation of $\check{F}(s)$, we have for $s \in V$ the equation

$$x(s)Q(x(s), 0) - f(x(s)) = 0.$$

The function $xQ(x, 0) - f(x)$ is meromorphic at $x = 0$, hence by Lemma 6.1.6 it is zero, so we have the equation

$$x^{d+1}Q(x, 0) = P(x).$$

But we have that $Q(x, 0) = 1 + O(x)$, which is a contradiction because $P(x)$ has degree at most d. □

6.2.2 Retrieving D-algebraic solutions

In the remaining cases, we are able to prove that $Q(x, y)$ is D-algebraic by lifting rational solutions to the decoupling equations ($E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega}$) and ($E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2}$) to algebraic solutions of (6.1.1).

Lemma 6.2.6. *Assume that $\omega = 0$, and that (h_1, h_2) is a nonzero solution to ($E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega}$) or ($E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2}$). Then the function h_2 satisfies the identity $\frac{h_2^{\sigma}(s)}{h_2(s)} = \frac{\widetilde{\gamma}^{\iota_1}(s)}{\widetilde{\gamma}(s)}$.*

Proof. Let (h_1, h_2) be a pair solution to either ($E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega}$) or ($E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2}$). In both cases, there exists some $\varepsilon \in \{-1, 1\}$ such that $h_1^{\iota_1}(s) = \varepsilon h_1(s)$ and $h_2^{\iota_2}(s) = \varepsilon h_2(s)$.

Analogously to what was done in the first section, we start from the identity

$$\widetilde{\gamma}(s)h_1(s) = -h_2(s). \tag{6.2.7}$$

Applying ι_1 on both sides of the equation and using the relation $h_1^{\iota_1} = \varepsilon h_1$, we obtain

$$\varepsilon\widetilde{\gamma}^{\iota_1}(s)h_1(s) = -h_2^{\iota_1}(s). \tag{6.2.8}$$

Eliminating $h_1(s)$ between (6.2.7) and (6.2.8), and using the identity $h_2^{\iota_1}(s) = \varepsilon h_2^{\iota_2\,\iota_1}(s) = \varepsilon h_2^{\sigma}(s)$ shows the claim. □

Lemma 6.2.7. *Assume that $\omega = 0$.*

1. *If ($E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega}$) admits a nonzero solution $(h_1, h_2) \in \mathbb{C}(s)$, then $Q(x, y)$ is rational in x and y (for the fixed t of Proposition 6.1.4). More precisely, there exist $H_1(z), H_2(z) \in \mathbb{C}(z)$ such that $H_1(x(s)) = h_1(s)$ and $H_2(y(s)) = h_2(s)$, and $\lambda \in \mathbb{C}$ such that*

$$xQ(x, 0) = \lambda H_1(x), \qquad yQ(0, y) = \lambda H_2(y).$$

2. *If ($E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2}$) admits a nonzero solution $(h_1, h_2) \in \mathbb{C}(s)$, then $Q(x, y)$ is algebraic over $\mathbb{C}(x, y)$ (for the fixed t of Proposition 6.1.4). More precisely, there exist $H_1(z), H_2(z) \in \mathbb{C}(z)$ such that $H_1(x(s)) = h_1(s)^2$ and $H_2(y(s)) = h_2(s)^2$, and $\lambda \in \mathbb{C}$ such that*

$$xQ(x, 0) = \pm\lambda\sqrt{H_1(x)}, \qquad yQ(0, y) = \pm\lambda\sqrt{H_2(y)}.$$

Proof. Let (h_1, h_2) be a nonzero solution to $(E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega})$ or $(E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2})$. By Lemma 6.2.6, the function $h_2(s)$ satisfies

$$\frac{h_2^{\sigma}(s)}{h_2(s)} = \frac{\widetilde{\gamma}^{\iota_1}(s)}{\widetilde{\gamma}(s)}. \tag{6.2.9}$$

Now, considering the function $H(s) \stackrel{\text{def}}{=} \frac{\widetilde{G}(s)}{h_2(s)}$ (recall that h_2 is nonzero), we see by combining equations (6.2.9) and (6.1.13) that $H(qs) = H(s)$. The function $H(s)$ is meromorphic on $\mathbb{C}$ and $|q| \neq 1$, hence $H(s)$ is a constant. Therefore, there exists $\lambda \in \mathbb{C}$ such that $\widetilde{G}(s) = \lambda h_2(s)$. By Equation (6.1.12), we also deduce that $\widetilde{F}(s) = \lambda h_1(s)$.

We can now prove the two cases of the lemma:

1. If $(h_1(s), h_2(s))$ is a solution to $(E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega})$, then by Proposition 6.1.2, there exist $H_1(x) \in \mathbb{C}(x)$ and $H_2(y) \in \mathbb{C}(y)$ such that $H_1(x(s)) = h_1(s)$ and $H_2(y(s)) = h_2(s)$. Therefore, $x(s)Q(x(s), 0) = \lambda H_1(x(s))$, and $y(s)Q(0, y(s)) = \lambda H_2(y(s))$ for all s in V. Thus, as these functions are meromorphic at $s = 0$, Proposition 6.1.6 yields $xQ(x, 0) = \lambda H_1(x)$ and $yQ(0, y) = \lambda H_2(y)$.
2. If $(h_1(s), h_2(s))$ is a solution to $(E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2})$, then $(h_1(s)^2)^{\iota_1} = h_1(s)^2$ and $(h_2(s)^2)^{\iota_2} = h_2(s)^2$. By Proposition 6.1.2, there exist $H_1(x) \in \mathbb{C}(x)$ and $H_2(y) \in \mathbb{C}(y)$ such that $H_1(x(s)) = h_1(s)^2$ and $H_2(y(s)) = h_2(s)^2$. Therefore, $x(s)^2Q(x(s), 0)^2 = \lambda^2 H_1(x(s))$ and $y(s)^2Q(0, y(s)^2 = \lambda^2 H_2(y(s))$ for all s in V. Thus, as these functions are meromorphic at $s = 0$, Proposition 6.1.6 yields $x^2Q(x, 0)^2 = \lambda^2 H_1(x)$ and $y^2Q(0, y)^2 = \lambda^2 H_2(y)$. □

Remark 6.2.8.
1. The value of λ can be found by evaluating $H_1(x)$ at $x = 0$ (resp. $H_2(y)$ at $y = 0$), and using the fact that $Q(0, 0) = 1$.
2. Note that the expressions for $H_1(x)$ and $H_2(y)$ depend *a priori* on the value of the real number t. However, in all the algebraic cases, the fractions $H_1(z)$ and $H_2(z)$ will be fixed fractions of $\mathbb{Q}(d_{i,j}, a, b, t, z)$ analytic at $t = 0$. Thus, we may lift the solutions for $Q(x, 0)$ and $Q(0, y)$ as formal power series in t.

■

The classification will go as follows: for every model and parameters, we will show that we are either in the case of application of Lemma 6.2.5 or Lemma 6.2.7, by studying the decoupling equations $(E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega})$ and $(E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2})$.

6.3 Decoupling equations

In the previous section, we reduced the classification to the study of two decoupling equations $(E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega})$ and $(E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2})$. According to the previously designed strategy, we will investigate both equations separately.

6.3.1 Homogeneous equation

We first handle the homogeneous equation $(E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2})$, which corresponds to the standard case of a *multiplicative decoupling* (like for instance those in [BEFHR25]), in this case of the fraction $\widetilde{\gamma} \stackrel{\text{def}}{=} \widetilde{\gamma}_1/\widetilde{\gamma}_2$. We want to determine in which cases this equation admits rational solutions. When they exist, we provide them explicitly. Otherwise, we provide two distinct arguments to show the non-existence of rational solutions. The first one is standard, and revolves around a process called *pole propagation*.

Lemma 6.3.1. *Let $u(s)$ be in $\mathbb{C}(s)$. Assume that there exists a pole (resp. zero) $P \neq \{0, \infty\}$ of $u(s)$ such that for all n in $\mathbb{Z}$ the point $\sigma^n P$ is never a zero (resp. pole) of $u(s)$.*

Then there is no nonzero $h(s) \in \mathbb{C}(s)$ such that $h^\sigma(s) = u(s)h(s)$.

Proof. The result is elementary, see for instance [HS08, Lemma 3.5] for a proof. □

We apply this lemma to the homogeneous equation ($E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2}$).

Corollary 6.3.2. *Assume that there exists $P \neq \{0, \infty\}$ a pole (resp. zero) of $\frac{\widetilde{\gamma}^{\iota_1}}{\widetilde{\gamma}}$ such that for all n in $\mathbb{Z}$ the point $\sigma^n P$ is never a zero (resp. pole) of $\frac{\widetilde{\gamma}^{\iota_1}}{\widetilde{\gamma}}$. Then the equation ($E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2}$) has no nonzero rational solution.*

Proof. Assume that (h_1, h_2) is a nonzero solution to ($E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2}$). Then by Lemma 6.2.6, the function $h_2(s)$ satisfies the equation $\frac{h_2^\sigma(s)}{h_2(s)} = \frac{\widetilde{\gamma}^{\iota_1}(s)}{\widetilde{\gamma}(s)}$, which by assumption and Lemma 6.3.1 has no nonzero rational solution, a contradiction. □

Unfortunately, this pole propagation technique does not discard all cases where the homogeneous solution has no solution. Mainly, it may happen that there exists a pair of fractions (h_1, h_2) satisfying the linear relation with $h_1'^{\iota_1}(s) = \pm h_1'(s)$ and $h_2'^{\iota_2}(s) = \pm h_2'(s)$ (we call such relaxed solutions *signed solutions*). In this case, Corollary 6.3.2 will not apply. However, when such solution with "wrong" signs for either $h_1'^{\iota_1}(s)/h_1'(s)$ or $h_2'^{\iota_2}(s)/h_2'(s)$ exists, we show that ($E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2}$) has no nonzero rational solution (with the "right" signs).

Lemma 6.3.3. *Let (h_1', h_2') be a nonzero pair satisfying the relation $\widetilde{\gamma}_1(s)h_1'(s) + \widetilde{\gamma}_2(s)h_2'(s) = 0$ with $(h_1')^{\iota_1} = \pm h_1'$ and $(h_2')^{\iota_2} = \pm h_2'$. If $(h_1')^{\iota_1} = h_1'$ or $(h_2')^{\iota_2} = h_2'$, then Equation ($E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2}$) has no nontrivial rational solution.*

Proof. Assume that (h_1', h_2') is such a pair, and let (h_1, h_2) be a pair of rational solutions to ($E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2}$). Then we have the equation

$$\frac{h_1}{h_1'} = \frac{h_2}{h_2'} =: u.$$

Now, by the symmetries of the $h_{1,2}$ and $h_{1,2}'$, we have that $(u^2)^{\iota_1} = (u^2)^{\iota_2} = u^2$. Therefore, the fraction u^2 is fixed by σ, hence $u^2 \in \mathbb{C}$, so $u \in \mathbb{C}$.

Assuming that $(h_1')^{\iota_1} = h_1'$, we obtain that $h_1^{\iota_1} = (uh_1')^{\iota_1} = uh_1' = h_1$. As h_1 satisfies also $h_1^{\iota_1} = -h_1$, we deduce that $h_1 = 0$, and thus that $h_2 = 0$.

Similarly, assuming that $(h_2')^{\iota_2} = h_2'$, we obtain that $(h_1, h_2) = (0, 0)$. □

6.3.2 Inhomogeneous equation

To handle the inhomogeneous equation, we will proceed as in Lemma 6.3.1 by proving a propagation lemma adapted to $(E_{\tilde{\gamma}_1,\tilde{\gamma}_2,\omega})$ (namely Lemma 6.3.6 below). Unlike the previous case, this lemma will only restrict the possible poles of h_1 and h_2 to a finite set of points of $\mathbb{P}^1$, for (h_1, h_2) a solution. In most cases, we will be able to show that one of $h_{1,2}$ must have its poles restricted to the set $\{0, \infty\}$, which is one of the requirements of Lemma 6.2.5 which shows non-D-algebraicity of $Q(x, y)$ in x and y.

Before we state and prove the propagation lemma for the inhomogeneous equation, we recall two easy facts on poles of rational maps on a curve. The first fact concerns the relations between the poles of two functions related by a linear equation.

Lemma 6.3.4. *Assume that $(h_1, h_2) \in \mathbb{C}(s)^2$ satisfies the relation $u_1h_1 + u_2h_2 + u_3 = 0$ for some u_1, u_2, u_3 in $\mathbb{C}(s)$. If P is a pole of h_1 not in $\{(u_1)_0, (u_2)_\infty, (u_3)_\infty\}$, then it is a pole of h_2.*

Proof. Assume that P is a pole of $h_1(s)$. Since it is not a zero of $u_1(s)$, it is a pole of $u_1(s)h_1(s)$. By the relation, P is a pole of $u_2(s)h_2(s) + u_3(s)$. Since P is not a pole of $u_3(s)$, it is a pole of $u_2(s)h_2(s)$, and because it is not a pole of $u_2(s)$, this implies that P is a pole of $h_2(s)$. □

The second standard fact concerns the poles of a function stable by automorphisms.

Lemma 6.3.5. *Let h be in $\mathbb{C}(s)$, and τ an automorphism of $\mathbb{P}^1$. If P is a pole (resp. zero) of h, then $\tau^{-1}P$ is a pole (resp. zero) of h^τ. In particular, if $h^\tau = \lambda h$ for some nonzero $\lambda \in \mathbb{C}$, then the set of poles (resp. zeros) of h is stable under the action of τ.*

Proof. Assume that P given by $s_0 \in \mathbb{P}^1$ is a zero of h. Then $h^\tau(\tau^{-1}(s_0)) = h(\tau(\tau^{-1}(s_0))) = h(s_0) = 0$. □

We now state the announced propagation lemma, specific to $(E_{\tilde{\gamma}_1,\tilde{\gamma}_2,\omega})$. It depends on the divisors of the coefficients of this equation. Recall from Proposition 6.1.3 that

$$(\tilde{\gamma}_1) = P_1 + P_2 - 0 - \infty, \qquad (\tilde{\gamma}_2) = P_3 + P_4 - 0 - \infty; \qquad (\star)$$

and that $\omega = 1 - A - B$ is a constant. We define four finite sets $\mathcal{L}_1^-, \mathcal{L}_1^+, \mathcal{L}_2^-, \mathcal{L}_2^+ \subset \mathbb{P}^1$ as follows:

$$\begin{aligned} \mathcal{L}_1^- &\overset{\text{def}}{=} \{P_1, P_2, \iota_2 P_3, \iota_2 P_4\}, & \mathcal{L}_1^+ &\overset{\text{def}}{=} \iota_1 \mathcal{L}_1^- = \{\iota_1 P_1, \iota_1 P_2, \sigma^{-1} P_3, \sigma^{-1} P_4\}, \\ \mathcal{L}_2^- &\overset{\text{def}}{=} \{\sigma P_1, \sigma P_2, \iota_2 P_3, \iota_2 P_4\}, & \mathcal{L}_2^+ &\overset{\text{def}}{=} \iota_2 \mathcal{L}_2^+ = \{\iota_1 P_1, \iota_1 P_2, P_3, P_4\}. \end{aligned} \qquad (6.3.1)$$

We call the elements of the sets $\mathcal{L}_{1,2}^{+,-}$ the *critical points* of equation $(E_{\tilde{\gamma}_1,\tilde{\gamma}_2,\omega})$.

Lemma 6.3.6 (Pole propagation). *Let $(h_1, h_2) \in \mathbb{C}(s)^2$ be a solution to $(E_{\tilde{\gamma}_1,\tilde{\gamma}_2,\omega})$.*

Let P be a pole of h_1 distinct from $0, \infty$.

(i) If $P \notin \mathcal{L}_1^-$, then $\sigma^{-1}P$ is a pole of h_1.

(ii) If $P \notin \mathcal{L}_1^+$, then σP is a pole of h_1.

Let P be a pole of h_2 distinct from $0, \infty$.

(i') If $P \notin \mathcal{L}_2^-$, then $\sigma^{-1}P$ is a pole of h_2.

(ii') If $P \notin \mathcal{L}_2^+$, then σP is a pole of h_2.

Proof. We prove (i). Let P be a pole of h_1 distinct from $0, \infty$ and not in $\mathcal{L}_1^-$. Because $P \notin \{P_1, P_2, 0, \infty\} = \{(\widetilde{\gamma}_1)_0, (\widetilde{\gamma}_2)_\infty, (\omega)_\infty\}$ (see ($\star$) above), this implies by Lemma 6.3.4 that P is also a pole of h_2. Now, $h_2^{\iota_2} = h_2$, hence $\iota_2 P$ is a pole of h_2 by Lemma 6.3.5. Now, $P \notin \{\iota_2 P_3, \iota_2 P_4, 0, \infty\}$ hence $\iota_2 P \notin \{P_3, P_4, 0, \infty\} = \{(\widetilde{\gamma}_2)_0, (\widetilde{\gamma}_1)_\infty, (\omega)_\infty\}$, so by Lemma 6.3.4 the point $\iota_2 P$ is a pole of h_1. Finally, $h_1^{\iota_1} = h_1$, hence $\sigma^{-1}P = \iota_1(\iota_2 P)$ is a pole of h_1.

We now prove (ii). Let P be a pole of h_1 distinct from $0, \infty$ and not in $\mathcal{L}_1^+$. Since $P \notin \mathcal{L}_1^+ = \iota_1 \mathcal{L}_1^-$, then $\iota_1 P \notin \mathcal{L}_1^-$, and $\iota_1 P \neq 0, \infty$. By (i), this implies that $\sigma^{-1}(\iota_1 P) = \iota_1(\sigma P)$ is a pole of h_1. As $h_1^{\iota_1} = h_1$, this implies that σP is a pole of h_1 by Lemma 6.3.5.

The proofs of (i') and (ii') are similar. □

Using the critical points, we may thus describe the possible poles of rational solutions (h_1, h_2) to $(E_{\widetilde{\gamma}_1, \widetilde{\gamma}_2, \omega})$.

Lemma 6.3.7. *Let $(h_1, h_2) \in \mathbb{C}(s)^2$ be a solution to $(E_{\widetilde{\gamma}_1, \widetilde{\gamma}_2, \omega})$. If P is a pole of h_1 distinct from 0 or ∞, then there exist two integers $m, n \geq 0$ such that $\sigma^{-m}P \in \mathcal{L}_1^-$ and $\sigma^n P \in \mathcal{L}_1^+$. Likewise, if P is a pole of h_2 distinct from 0 or ∞, then there exist two integers $m, n \geq 0$ such that $\sigma^{-m}P \in \mathcal{L}_2^-$ and $\sigma^n P \in \mathcal{L}_2^+$.*

Proof. Assume for the sake of contradiction that $P \neq 0, \infty$ is a pole of h_1 satisfying $\sigma^n P \notin \mathcal{L}_1^+$ for all $n \geq 0$. Then by induction and Lemma 6.3.6, we show that $\sigma^n P$ is a pole of h_1 for all $n \geq 0$. Since P is distinct from 0 or ∞, the orbit $(\sigma^n P)_{n \geq 0}$ is infinite, hence the fraction h_1 has an infinite number of poles, a contradiction. The other points are proved in a similar fashion. □

6.3.3 σ-distance

Thanks to pole propagation, Lemma 6.3.7 allows us to locate the possible poles of h_1 and h_2 for a rational solution (h_1, h_2) to $(E_{\widetilde{\gamma}_1, \widetilde{\gamma}_2, \omega})$. Similarly, Lemma 6.3.1 gives a sufficient condition for proving that $(E'_{\widetilde{\gamma}_1, \widetilde{\gamma}_2})$ has no nonzero rational solution (h'_1, h'_2). These two lemmas thus give conditions to the existence of solutions to decoupling equations based on the relations between the points of the finite sets $\mathcal{L}_{1,2}^{\pm}$. These relations are captured by a signed distance called the *σ-distance*, that we introduce to compare two points of $\mathbb{P}^1$ with respect to the action of the group $< \sigma >$. This will give us numerical data from which we will conduct the classification.

Definition/Proposition 6.3.8. Let P and P' be two points of $\mathbb{P}^1$ distinct from 0 and ∞. We define the *σ-distance* $\delta(P, P')$ of the points P and P' as follows:

— If there exists an integer $n \in \mathbb{Z}$ such that $\sigma^n P = P'$, then n is unique, and we define $\delta(P, P') = n$.

— Otherwise, if no such integer exists, we define $\delta(P, P') = \perp$.

Proof. We just need to show the uniqueness. Assume that $\sigma^n P = \sigma^m P$ for two integers m and n. Then $\sigma^{n-m} P = P$, which is possible if and only if if $n = m$ because 0 and ∞ are the only periodic points of the action of σ on $\mathbb{P}^1$ (indeed, if $q^n s = s$ with $n \geq 1$, then $s = 0$ or $s = \infty$, for $q^n \neq 1$). □

The σ-distance satisfies the standard arithmetic properties that one would expect from a signed distance.

Proposition 6.3.9. *With the convention that $n + \perp = \perp$ for every integer n, and that $\perp = -\perp$, the σ-distance satisfies the following properties for points P, P' and P'' distinct from 0 or ∞:*

(i) $\delta(P, P') = -\delta(P', P)$,

(ii) $\delta(P, P') + \delta(P', P'') = \delta(P, P'')$ *if* $\delta(P, P')$ *and* $\delta(P', P'')$ *are finite,*

(iii) $\delta(P, \sigma(P')) = \delta(P, P') + 1$,

(iv) $\delta(P, P') = \delta(\iota_1 P', \iota_1 P) = \delta(\iota_2 P', \iota_2 P)$.

Proof. The proofs of (i), (ii) and (iii) are straightforward, hence we focus on the proof of (iv). Assume first that $\sigma^n P = P'$ for some integer n. Recall that $\sigma = \iota_2 \iota_1$ with $\iota_1^2 = \iota_2^2 = \mathrm{id}$, so it is easy to see that $\iota_1 \sigma^n = \sigma^{-n} \iota_1$. Thus, $\sigma^{-n}(\iota_1 P) = \iota_1(\sigma^n P) = \iota_1 P'$, which implies that $\delta(\iota_1 P, \iota_1 P')$ is finite, being equal to $-n = -\delta(P, P') = \delta(P', P)$. The application ι_1 is an involution, thus if $\delta(P, P') = \perp$, then we also have $\delta(\iota_1 P, \iota_1 P') = \perp$. □

We are going to determine the values of $\delta(P, P')$ for all $(P, P') \in \mathcal{L}_1^- \times \mathcal{L}_1^+$ and $(P, P') \in \mathcal{L}_2^- \times \mathcal{L}_2^+$. These values are compiled respectively in the matrices M_1 and M_2 in $\mathcal{M}_4(\mathbb{Z} \cup \{\perp\})$, the lines ($\mathcal{L}_{1,2}^-$) and columns ($\mathcal{L}_{1,2}^+$) being ordered as in (6.3.1). More precisely, their entries are organized as follows:

$$
\begin{aligned}
M_1 &\stackrel{\text{def}}{=} \begin{pmatrix} \delta(P_1, \iota_1 P_1) & \delta(P_1, \iota_1 P_2) & \delta(P_1, \sigma^{-1} P_3) & \delta(P_1, \sigma^{-1} P_4) \\ \delta(P_2, \iota_1 P_1) & \delta(P_2, \iota_1 P_2) & \delta(P_2, \sigma^{-1} P_3) & \delta(P_2, \sigma^{-1} P_4) \\ \delta(\iota_2 P_3, \iota_1 P_1) & \delta(\iota_2 P_3, \iota_1 P_2) & \delta(\iota_2 P_3, \sigma^{-1} P_3) & \delta(\iota_2 P_3, \sigma^{-1} P_4) \\ \delta(\iota_2 P_4, \iota_1 P_1) & \delta(\iota_2 P_4, \iota_1 P_2) & \delta(\iota_2 P_4, \sigma^{-1} P_3) & \delta(\iota_2 P_4, \sigma^{-1} P_4) \end{pmatrix}, \\
M_2 &\stackrel{\text{def}}{=} \begin{pmatrix} \delta(\sigma P_1, \iota_1 P_1) & \delta(\sigma P_1, \iota_1 P_2) & \delta(\sigma P_1, P_3) & \delta(\sigma P_1, P_4) \\ \delta(\sigma P_2, \iota_1 P_1) & \delta(\sigma P_2, \iota_1 P_2) & \delta(\sigma P_2, P_3) & \delta(\sigma P_2, P_4) \\ \delta(\iota_2 P_3, \iota_1 P_1) & \delta(\iota_2 P_3, \iota_1 P_2) & \delta(\iota_2 P_3, P_3) & \delta(\iota_2 P_3, P_4) \\ \delta(\iota_2 P_4, \iota_1 P_1) & \delta(\iota_2 P_4, \iota_1 P_2) & \delta(\iota_2 P_4, P_3) & \delta(\iota_2 P_4, P_4) \end{pmatrix}.
\end{aligned}
\tag{6.3.2}
$$

Proposition 6.3.10. *The matrices M_1 and M_2 satisfy the following relations:*

(i) $M_1^T = M_1$ *and* $M_2^T = M_2$,

(ii) $M_2 = M_1 + \begin{pmatrix} -J_2 & 0 \\ 0 & J_2 \end{pmatrix}$ *where* $J_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

Proof. These are straightforward applications of Proposition 6.3.9. □

Thanks to the above proposition, it is only required to compute the σ-distances of 10 pairs of points, namely those above the diagonal of M_1. Note that the sets $\mathcal{L}^{\pm}_{1,2}$, and thus the matrix M_1 depend on the set of steps $\mathcal{S}_i$ of the model and weights $d_{i,j}$, A and B. We finish this subsection by giving two lemmas to exploit these matrices.

Lemma 6.3.11. *Let (h_1, h_2) be a pair of rational solutions to $(E_{\tilde{\gamma}_1,\tilde{\gamma}_2,\omega})$.*

(i) If all the entries of M_1 are in $\mathbb{Z}^- \cup \{\perp\}$, then the poles of $h_1(s)$ belong to $\{0, \infty\}$.

(ii) If all the entries of M_2 are in $\mathbb{Z}^- \cup \{\perp\}$, then the poles of $h_2(s)$ belong to $\{0, \infty\}$.

Proof. Let (h_1, h_2) be a pair of rational solutions to $(E_{\tilde{\gamma}_1,\tilde{\gamma}_2,\omega})$. We prove (i). Assume that all the entries of M_1 are in $\mathbb{Z}^- \cup \{\perp\}$. If $P \neq \{0, \infty\}$ is a pole of h_1, then by Lemma 6.3.7, there exist $m, n \geq 0$ such that $\sigma^{-m}P =: Q^- \in \mathcal{L}^-_1$ and $\sigma^n P =: Q^+ \in \mathcal{L}^+_1$. But then by (ii) of Proposition 6.3.9, one has $\delta(Q^-, Q^+) = \delta(Q^-, P) + \delta(P, Q^+) = n + m \geq 0$, a contradiction since $\delta(Q^-, Q^+)$ is an entry of M_1. Thus, h_1 has no poles besides 0 and ∞. The proof of point (ii) is similar. □

Lemma 6.3.12. *If one of the rows of M_1 consists of $\perp$'s only, then $(E'_{\tilde{\gamma}_1,\tilde{\gamma}_2})$ has no nonzero rational solution.*

Proof. From Proposition 6.1.3, we may write the divisor of $\tilde{\gamma}^{\iota_1}/\tilde{\gamma}$ as

$$(\tilde{\gamma}^{\iota_1}/\tilde{\gamma}) = \iota_1 P_1 + \iota_1 P_2 + P_3 + P_4 - P_1 - P_2 - \iota_1 P_3 - \iota_1 P_4.$$

Assume that there exists $Q^- \in \mathcal{L}^-_1$ such that for all $Q^+ \in \mathcal{L}^+_1$ one has $\delta(Q^-, Q^+) = \perp$ (Q^- labels the row M_1 consisting of $\perp$'s only). Then there exists an integer k such that $Q' \overset{\text{def}}{=} \sigma^k Q^-$ is a pole of $\tilde{\gamma}^{\iota_1}/\tilde{\gamma}$, namely $k = 0$ for $Q^- \in \{P_1, P_2\}$ or $k = -1$ for $Q^- \in \{\iota_2 P_3, \iota_2 P_4\}$.

— The point Q' is a pole of $\tilde{\gamma}^{\iota_1}/\tilde{\gamma}$.

— The point $\sigma^n Q'$ is never a zero of $\tilde{\gamma}^{\iota_1}/\tilde{\gamma}$. Indeed, if it were the case, then $Q^+ \overset{\text{def}}{=} \sigma^m Q'$ would belong to $\mathcal{L}^+_1$, either for $m = n$ if $\sigma^n Q' \in \{\iota_1 P_1, \iota_1 P_2\}$, or $m = n - 1$ if $\sigma^n Q' \in \{P_3, P_4\}$. The point (iii) of Proposition 6.3.9 would then imply that

$$\delta(Q^-, Q^+) = \delta(Q^-, Q') + \delta(Q', Q^+) = k + m,$$

while the row of M_1 corresponding to Q^- consisting of $\perp$'s only implies that $\delta(Q^-, Q^+) = \perp$, a contradiction.

Therefore by Corollary 6.3.2, $(E'_{\tilde{\gamma}_1,\tilde{\gamma}_2})$ has no nonzero rational solution. □

6.4 Computing the σ-distance

Denote by $\mathbb{F}$ the field $\mathbb{Q}(d_{i,j}, a, b)$. In this section, we describe a heuristic to decide given two points P and P' of $\mathbb{P}^1$ if there exists an integer n such that $\sigma^n(P) = P'$. In other words, for two points P and P' of $\mathbb{P}^1$, the goal is to compute the σ-distance $\delta(P, P')$ of Definition 6.3.8. When we restrict to the points that originate from zeros of the fractions $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$, the σ-distance is computable. The main argument used here, and already exploited for instance in [BM10], is based on *valuations*.

Definition/Proposition 6.4.1. There exists an embedding (a $\mathbb{F}$-algebra homomorphism) $\psi : \overline{\mathbb{F}(t)} \longrightarrow \mathbb{C}^{frac}((T))$ where $\mathbb{C}^{frac}((T))$ is the field of formal Puiseux series over $\mathbb{C}$ in the formal variable T.* We fix once and for all this embedding ψ. As a result, for any $u \in \overline{\mathbb{F}(t)}$, we define its *valuation* $v(u)$ to be the valuation in the variable T of $\psi(u)$ (the valuation of 0 being $+\infty$).

Proof. Consider the embedding $\psi_5 : \mathbb{F}(t) \longrightarrow \mathbb{C}^{frac}((T))$ defined as the composition

$$\mathbb{F}(t) \xrightarrow{\psi_1} \mathbb{F}(T) \xrightarrow{\psi_2} \mathbb{C}(T) \xrightarrow{\psi_3} \mathbb{C}((T)) \xrightarrow{\psi_4} \mathbb{C}^{frac}((T))$$

where the embedding $\psi_1 : t \mapsto T$ is the isomorphism between $\mathbb{F}(t)$ and $\mathbb{F}(T)$ as t is transcendental over $\mathbb{F}$; the embedding ψ_2 is the map induced by the inclusion $\mathbb{F} \subset \mathbb{C}$; the embedding ψ_3 is the map from $\mathbb{C}(T)$ into the field of Laurent series $\mathbb{C}((T))$; and the embedding ψ_4 is an arbitrary embedding of $\mathbb{C}((T))$ into $\mathbb{C}^{frac}((T))$ its algebraic closure (this follows from the Newton-Puiseux theorem since $\mathbb{C}$ is algebraically closed of characteristic zero [Sta24, Th. 6.1.5]). As the field $\mathbb{C}^{frac}((T))$ is algebraically closed, the embedding ψ_5 admits an extension ψ to an embedding of $\overline{\mathbb{F}(t)}$ into $\mathbb{C}^{frac}((T))$ ([Lan02, Th. V.2.8]). □

Definition 6.4.2. Let $P \in \mathbb{P}^1 \setminus \{0, \infty\}$ such that $\phi(P) = ([1 : x_1], [1 : y_1]) \in \mathbb{P}^1 \times \mathbb{P}^1$ with $x_1, y_1 \in \overline{\mathbb{F}(t)}$. Then define the *bivaluation* of P to be $v(P) \overset{\text{def}}{=} (v(x_1), v(y_1))$.

Lemma 6.4.3. *Let $H(x,y) \in \mathbb{F}(t)(x,y)$ be a fraction such that $h(s) \overset{\text{def}}{=} H(x(s), y(s)) \in \mathbb{C}(s)$ is well defined. If $P \in \mathbb{P}^1 \setminus \{0,\infty\}$ is a pole or zero of h, then the point $\phi(P) = ([1 : x_1], [1 : y_1]) \in \overline{E_t} \subset \mathbb{P}^1 \times \mathbb{P}^1$ is distinct from $\Omega = (0,0)$, and x_1 and y_1 are algebraic over $\mathbb{F}(t)$.*

Proof. Let s_0 be the coordinate in $\mathbb{P}^1$ of P. By assumption, $s_0 \neq 0, \infty$, hence Proposition 6.1.1 yields $x(s_0) \neq 0$ and $y(s_0) \neq 0$. Moreover, the functions $x(s)$ and $y(s)$ belong to $\overline{\mathbb{F}(t)}(s)$. Hence, $h(s) \in \overline{\mathbb{F}(t)}(s)$, thus if s_0 is a pole or zero of $h(s)$, then s_0 belongs to $\overline{\mathbb{F}(t)}$. Therefore, so do $x_1 = \frac{1}{x(s_0)}$ and $y_1 = \frac{1}{y(s_0)}$. □

The above lemma allows us to talk about the bivaluations of the zeros of $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ (the points P_i defined in Proposition 6.1.3). Now, recall the expressions for ι_1 and ι_2 on $\overline{E_t} \subset \mathbb{P}^1 \times \mathbb{P}^1$ of (6.1.4) and (6.1.5):

$$\begin{aligned} \iota_1([1:x_1],[1:y_1]) &= \left([1:x_1], \left[1 : \frac{d_{-1,1}x_1^2 + d_{0,1}x_1 + d_{1,1}}{d_{1,-1}y_1}\right]\right), \\ \iota_2([1:x_1],[1:y_1]) &= \left(\left[1 : \frac{d_{1,-1}y_1^2 + d_{1,0}y_1 + d_{1,1}}{d_{-1,1}x_1}\right], [1:y_1]\right). \end{aligned} \tag{6.4.1}$$

Hence, for a point $P \in \mathbb{P}^1 \setminus \{0,\infty\}$, the homogeneous coordinates at infinity of $\phi(\iota_1 P) = \iota_1(\phi(P))$ and $\phi(\iota_2 P) = \iota_2(\phi(P))$ (Proposition 6.1.1) are explicit rational functions in the coordinates of $\phi(P)$. Hence, if the coordinates x_1 and y_1 of $\phi(P)$ are in $\overline{\mathbb{F}(t)}$,

*. Although the real number t is transcendental over $\mathbb{F}$, we do not directly consider the field of formal Puiseux series in t to avoid conflicts of notation with the usual sum of complex numbers.

then so are the coordinates of $\phi(\iota_1 P)$ and $\phi(\iota_2 P)$. Thus, all the points of the sets $\mathcal{L}^{\pm}_{1,2}$ defined in (6.3.1) admit a bivaluation. This raises the possibility of keeping track of the successive bivaluations of the points $\sigma^n(P)$ for all integers n and $P \in \mathcal{L}^{\pm}_{1,2}$. It turns out that in most cases, the bivaluation of $\sigma(P)$ depends only on the bivaluation of P.

Lemma 6.4.4. *Let P be in $\mathbb{P}^1$ with $\phi(P) = ([1:x_1],[1:y_1])$ and $x_1, y_1 \in \overline{\mathbb{F}(t)}$, and let $v(P) = (i,j)$.*

(1) If $i < 0$ then $v(\iota_1(P)) = (i, 2i - j)$.

(2) If $j < 0$ then $v(\iota_2(P)) = (2j - i, j)$.

Proof. We show (1). Write $\iota_1([1:x_1],[1:y_1]) = ([1:x_1],[1:y_1'])$ (as read in (6.4.1)). If $v(x_1) < 0$, then since $d_{-1,1} \neq 0$, we have $v(d_{-1,1}x_1^2) = 2v(x_1) = 2i$ and $v(d_{0,1}x_1 + d_{1,1}) \geq v(x_1) > 2v(x_1) = 2i$. Thus, the numerator of y_1' has valuation $2i$. Moreover, since $d_{1,-1} \neq 0$, we compute the valuation of the denominator of y_1' as $v(d_{1,-1}y_1) = v(y_1) = j$, hence the result. The proof of (2) is similar. □

Lemma 6.4.5. *Let $P \in \mathbb{P}^1$ with $\phi(P) = ([1:x_1],[1:y_1])$ and $x_1, y_1 \in \overline{\mathbb{F}(t)}$, and let $v(P) = (i,j)$ and $\delta = |i - j|$.*

(1) If $i < j < 0$ then $v(\sigma^k(P)) = (i - 2\delta k, j - 2\delta k)$ for all $k \geq 0$,

(2) If $j < i < 0$ then $v(\sigma^{-k}(P)) = (i - 2\delta k, j - 2\delta k)$ for all $k \geq 0$.

Proof. We prove (1). Assume that $\phi(P) = ([1:x_1],[1:y_1])$ with $x_1, y_1 \in \overline{\mathbb{F}(t)}$ and $(i,j) \overset{\text{def}}{=} v(P) < 0$. We first compute the bivaluation of $\iota_1 P$. As $i < 0$, Lemma 6.4.4 asserts that $v(\iota_1(P))$ is completely determined by i and j. Thus,

$$v(\iota_1 P) = (i, 2i - j) = (i, j + 2(i - j)) = (i, j - 2\delta) \text{ since } i < j.$$

As $j - 2\delta < 0$, the bivaluation of $\sigma(P) = \iota_2(\iota_1 P)$ is also completely determined by i and j by Lemma 6.4.4, and thus

$$v(\sigma P) = (2(j - 2\delta) - i, j - 2\delta) = (3i - 2j, j - 2\delta) = (i - 2\delta, j - 2\delta).$$

We thus proved that if $v(P) = (i,j)$ with $i < j < 0$, then $v(\sigma P) = (i - 2\delta, j - 2\delta)$. An easy induction completes the proof. The proof of the second point is similar. □

As the classification of the nature of $Q(x,y)$ given a weighted model depends on the matrices M_1 and M_2 defined in (6.3.2), we need to compute $\delta(Q^-, Q^+)$ for two points $Q^- \in \mathcal{L}_1^-$ and $Q^+ \in \mathcal{L}_1^+$ for these weights $d_{i,j}$, a, b. Thus, we are able to determine the tables of Section 6.4. Note that from Lemma 6.4.3, both points Q^- and Q^+ have a bivaluation.

1. Compute $\sigma^n Q^-$ for $n \in \{-3, -2, -1, 0, 1\}$. It happens in all cases (when $Q^- \in \mathcal{L}_1^-$) that $v(\sigma^{-2}(Q^-)) = (i,j)$ with $j < i < 0$ and $v(\sigma^2(Q^-)) = (i,j)$ with $i < j < 0$. Thus, Lemma 6.4.5 allows us to determine the sequence of bivaluations of $\sigma^n(Q^-)$ for all $n \in \mathbb{Z}$.

2. Determine the bivaluation (i', j') of the point Q^+.
   - If one point of the orbit of Q^- has bivaluation (i', j') (which we may decide, see the above point), then there are a finite number of n such that $v(\sigma^n(Q^-)) = (i', j')$. For each of these n, check if $\sigma^n(Q^-) = Q^+$. If one of these n works, then $\delta(Q^-, Q^+) = n$.
   - Otherwise, if the bivaluation (i', j') does not appear in the orbit of Q^- or no n works, then $\delta(Q^-, Q^+) = \perp$.

Of course, the space of parameters a, b, $d_{i,j}$ is infinite. Hence, the actual procedure adds the following level of complexity: the bivaluation of a point Q depends on an algebraic condition on the parameters. Thus, we need to explore all the possible bivaluations according to these parameters (for the points Q^- and Q^+). The core of the procedure stays the same. Instead of giving a dry algorithm, we expand below an example.

Example 6.4.6. We consider the set of steps $\mathcal{S}_1$ of Figure 6.0.1. In this example, we show how to construct the entry $\delta(P_2, \iota_1 P_2)$ of Table 6.4.1, depending on the weights $d_{0,1}$, $d_{1,-1}$, $d_{-1,1}$, A and B.

1. The first step consists in computing the bivaluation of P_2 depending on the weights $d_{i,j}$, A and B. We first compute the coordinates of $\phi(P_2)$ for generic $d_{i,j}$, A and B:

$$\phi(P_2) = \left(\left[1 : \frac{-d_{0,1}d_{1,-1}t^2}{d_{1,-1}d_{-1,1}t^2 + A^2 - A} \right], \left[1 : \frac{-Ad_{0,1}t}{d_{1,-1}d_{-1,1}t^2 + A^2 - A} \right] \right).$$

   The Laurent series expansions of the coordinates of P_2 at $t = 0$ are as follows

$$\phi(P_2) = \left(\left[1 : \frac{-d_{0,1}d_{1,-1}t^2}{A(A-1)} + O(t^3) \right], \left[1 : \frac{-Ad_{0,1}t}{A(A-1)} + O(t^2) \right] \right).$$

   We thus notice that when $A \neq 0$, the bivaluation of P_2 is $v(P_2) = (2, 1)$ (the weights $d_{i,j}$ are nonzero, and note that as $A = 1 - \frac{1}{a}$, then A cannot be equal to 1). Otherwise, in the case $A = 0$, we find $v(P_2) = (0, \infty)$. We now compute their orbits in these two cases.

   (a) Assume that $A \neq 0$, so that $v(P_2) = (2, 1)$. We check by computing the points $(\sigma^i P_2)_{-2 \le i \le 2}$ that their bivaluations do not depend on A as long at it is nonzero, nor the weights $d_{1,-1}$, $d_{-1,1}$ and $d_{0,1}$, and that they are equal to

$$\cdots \to_\sigma (-2, -3) \to_\sigma (0, -1) \to_\sigma v(P_2) = (2, 1) \to_\sigma (0, 1) \to_\sigma (-2, -1) \to_\sigma \ldots . \tag{6.4.2}$$

   The remaining parts of the above sequence may be continued by the use of Lemma 6.4.5. Indeed, $v(\sigma^{-2} P_2) = (-2, -3)$, hence $\sigma^{-2} P_2$ satisfies condition (2) of Lemma 6.4.5, hence we know that whatever the value of $A \neq 0$, one has $v(\sigma^{-k-2} P_2) = (-2 - 2k, -3 - 2k)$. Similarly, $v(\sigma^2 P_2) = (-2, -1)$ satisfies condition (1) of Lemma 6.4.5, hence we deduce that $v(\sigma^{k+2} P_2) = (-2 + 2k, -1 + 2k)$ regardless of the values of the weights $d_{i,j}$ and $A \neq 0$.

(b) For $A = 0$, using the same technique, we compute the sequence of bivaluations for $(\sigma^i P_2)_{-3\leq i\leq 1}$ and $A = 0$ (with $v(P_2) = (0, \infty)$):

$$\cdots \to_\sigma (-2,-3) \to_\sigma (0,-1) \to_\sigma (\infty,\infty) \to_\sigma v(P_2) = (0,\infty) \to_\sigma (-2,-1) \to_\sigma \ldots, \tag{6.4.3}$$

Again, the remaining of the above sequence may be continued using Lemma 6.4.5.

2. We now compute $\delta(P_2, \iota_1 P_2)$, for $A = 0$ and $A \neq 0$. We first compute the coordinates of $\phi(\iota_1 P_2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ for generic values of the weights $d_{i,j}$, A and B:

$$\phi(\iota_1 P_2) = \left(\left[1 : -\frac{d_{0,1}d_{1,-1}t^2}{d_{1,-1}d_{-1,1}t^2 + A^2 - A} \right], \left[1 : \frac{d_{0,1}t(A-1)}{d_{1,-1}d_{-1,1}t^2 + A^2 - A} \right] \right).$$

We find that

$$v(\iota_1 P_2) = \begin{cases} (2,1) & \text{if } A \neq 0 \\ (0,-1) & \text{if } A = 0 \end{cases}.$$

(a) For $A \neq 0$, since $v(\iota_1 P_2) = (2,1)$, we see by looking at (6.4.2) that if $\iota_1 P_2$ belongs to the orbit of P_2, then $\iota_1 P_2 = P_2$. This condition is satisfied if and only if $A = \frac{1}{2}$, and then $\delta(P_2, \iota_1 P_2) = 0$. Otherwise, $\delta(P_2, \iota_1 P_2) = \perp$.

(b) For $A = 0$, since $v(\iota_1 P_2) = (0,-1)$, we see by looking at (6.4.3) that if $\iota_1 P_2$ belongs to the orbit of P_2, then $\iota_1 P_2 = \sigma^{-2} P_2$. This is always the case for any weighting $d_{i,j}$, thus $\delta(P_2, \iota_1 P_2) = -2$.

We thus compute the corresponding entry of Table 6.4.1

$$\delta(P_2, \iota_1 P_2) = \begin{cases} 0 & \text{if } A = \frac{1}{2} \\ -2 & \text{if } A = 0 \\ \perp & \text{otherwise.} \end{cases}$$

The other entries are computed in the same way. ∎

Using these results, we manage to compute M_1 as defined in (6.3.2). For clarity, the genus 0 models are subdivided according to their underlying set of steps, and there are thus five different tables, that are 6.4.1, 6.4.2, 6.4.3, 6.4.4 and 6.4.5. For each table, the value of every entry depends algebraically on the complex parameters $A = 1 - \frac{1}{a}$, $B = 1 - \frac{1}{b}$ and $d_{i,j}$.

Note that from Proposition 6.3.10, the matrix M_1 is symmetric so only the entries on the upper diagonal are specified, and a simple computation allows us to deduce the entries of M_2 from those of M_1. Also note that for each table, the labels P_1, P_2 of the zeros of of $\widetilde{\gamma}_1$ are chosen arbitrarily and fixed once and for all for each model. The same goes for the labels P_3, P_4 of the zeros of $\widetilde{\gamma}_2$,

6.5 Classification

We are now geared to prove the classification.

	$\iota_1 P_1$	$\iota_1 P_2$	$\sigma^{-1} P_3$	$\sigma^{-1} P_4$
P_1	0	-1 if $A = 0$ $\perp$ otherwise	-1	-1 if $B = 0$ $\perp$ otherwise
P_2		0 if $A = \frac{1}{2}$ -2 if $A = 0$ $\perp$ otherwise	-2 if $A = 0$ $\perp$ otherwise	0 if $A + B = 1$ -2 if $A = B = 0$ $\perp$ otherwise
$\iota_2 P_3$			-2	-2 if $B = 0$ $\perp$ otherwise
$\iota_2 P_4$				0 if $B = \frac{1}{2}$ -2 if $B = 0$ $\perp$ otherwise

Table 6.4.1 – Set of steps $\mathcal{S}_1$

	$\iota_1 P_1$	$\iota_1 P_2$	$\sigma^{-1} P_3$	$\sigma^{-1} P_4$
P_1	$\perp$	-1 if $A = 0$ $\perp$ otherwise	-1	$\perp$
P_2		$\perp$	$\perp$	0 if $A + B = 1$ -2 if $A = B = 0$ $\perp$ otherwise
$\iota_2 P_3$			$\perp$	-2 if $B = 0$ $\perp$ otherwise
$\iota_2 P_4$				$\perp$

Table 6.4.2 – Set of steps $\mathcal{S}_2$

	$\iota_1 P_1$	$\iota_1 P_2$	$\sigma^{-1} P_3$	$\sigma^{-1} P_4$
P_1	0 if $A = \frac{1}{2}$ $\perp$ otherwise	-1 if $A = 0$ $\perp$ otherwise	$\perp$	$\perp$
P_2		0 if $A = \frac{1}{2}$ $\perp$ otherwise	$\perp$	$\perp$
$\iota_2 P_3$			-1 if $B = \frac{1}{2}$ $\perp$ otherwise	-2 if $B = 0$ $\perp$ otherwise
$\iota_2 P_4$				-1 if $B = \frac{1}{2}$ $\perp$ otherwise

Table 6.4.3 – Set of steps $\mathcal{S}_3$

6.5.1 Some decouplings and homogeneous solutions

We first determine for which supports and weights the functional equation ($E'_{\tilde{\gamma}_1,\tilde{\gamma}_2}$) admits a nonzero rational solution. The two following computational lemmas will be used to exhibit particular signed solutions to ($E'_{\tilde{\gamma}_1,\tilde{\gamma}_2}$). These statements can be checked

	$\iota_1 P_1$	$\iota_1 P_2$	$\sigma^{-1} P_3$	$\sigma^{-1} P_4$
P_1	0 if $A = \frac{1}{2}$ -1 if (C_4) $\perp$ otherwise	-1 if $A = 0$ $\perp$ otherwise	$\perp$	$\perp$
P_2		0 if $A = \frac{1}{2}$ -1 if (C_4) $\perp$ otherwise	$\perp$	$\perp$
$\iota_2 P_3$			$\perp$	-2 if $B = 0$ $\perp$ otherwise
$\iota_2 P_4$				$\perp$

where $(C_4) \equiv (A = 0) \wedge (4d_{1,-1}d_{1,1} = {d_{0,1}}^2)$.

Table 6.4.4 – Set of steps $\mathcal{S}_4$

	$\iota_1 P_1$	$\iota_1 P_2$	$\sigma^{-1} P_3$	$\sigma^{-1} P_4$
P_1	-1 if (C_5) $\perp$ otherwise	-1 if $A = 0$ $\perp$ otherwise	$\perp$	$\perp$
P_2		-1 if (C_5) $\perp$ otherwise	$\perp$	$\perp$
$\iota_2 P_3$			-2 if (C_5') $\perp$ otherwise	-2 if $B = 0$ $\perp$ otherwise
$\iota_2 P_4$				-2 if (C_5') $\perp$ otherwise

where $(C_5) \equiv (A = 0) \wedge (4d_{1,1}d_{-1,1} = {d_{0,1}}^2)$ and $(C_5') \equiv (B = 0) \wedge (4d_{1,1}d_{-1,1} = {d_{1,0}}^2)$.

Table 6.4.5 – Set of steps $\mathcal{S}_5$

in the joint Maple worksheet. Recall the definitions of $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ in (6.1.6).

Lemma 6.5.1. *Assume that $d_{1,1} = 0$ (supports $\mathcal{S}_1$ and $\mathcal{S}_2$). Write $x = x(s)$ and $y = y(s)$, and let $\lambda \in \mathbb{C}$. Then the following identities hold:*

(i) $u_\lambda \stackrel{\text{def}}{=} (1-\lambda) - td_{1,0}x - td_{1,-1}\frac{x}{y} = -(\lambda - td_{0,1}y - td_{-1,1}\frac{y}{x})$,

(ii) $(\lambda - A + x\widetilde{\gamma}_1)\, u_\lambda = \lambda(1-\lambda) - t^2 d_{1,-1}d_{-1,1} - (\lambda t d_{1,0} + t^2 d_{1,-1}d_{0,1})x$,

(iii) $-(1 - \lambda - B + y\widetilde{\gamma}_2)\, u_\lambda = \lambda(1-\lambda) - t^2 d_{1,-1}d_{-1,1} - ((1-\lambda)td_{0,1} + t^2 d_{-1,1}d_{1,0})y$.

Lemma 6.5.2. *Assume that $d_{1,0} = d_{0,1} = 0$ (support $\mathcal{S}_3$). Write $x = x(s)$ and $y = y(s)$. Then the following identities hold:*

(i) $(\frac{1}{2} - A + x\widetilde{\gamma}_1)^2 = \frac{1}{4} - d_{1,-1}d_{-1,1}t^2 - d_{1,1}d_{1,-1}t^2x^2$,

(ii) $(\frac{1}{2} - B + y\widetilde{\gamma}_2)^2 = \frac{1}{4} - d_{1,-1}d_{-1,1}t^2 - d_{1,1}d_{-1,1}t^2y^2$.

Using these computations along with Lemma 6.3.3, and the tables of Section 6.4

along with Lemma 6.3.12, we build below Table 6.5.1 † which describes the cases where the homogeneous equation $(E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2})$ has solutions. Its entries are read as follows:

— Either the entry has the form $(\perp, i)$, meaning that row i of M_1 is made of $\perp$'s. In this case, the application of Lemma 6.3.12 shows that there is no nonzero solution to $(E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2})$.

— Either the entry has the form $(\varepsilon_1, \varepsilon_2) \in (\pm, \pm)$, which means that there is a nonzero pair of fractions (h'_1, h'_2) satisfying the relation $\widetilde{\gamma}_1(s)h'_1(s) + \widetilde{\gamma}_2(s)h'_2(s) = 0$ with $(h'_1)^{\iota_1}/h'_1 = \varepsilon_1$ and $(h'_2)^{\iota_2}/h'_2 = \varepsilon_2$ (a *signed solution* to $(E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2})$). If $(\varepsilon_1, \varepsilon_2) \neq (-,-)$, then the application of Lemma 6.3.3 shows that there is no nonzero solution to $(E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2})$.

Hence, Equation $(E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2})$ has a nonzero solution if and only if the entry of Table 6.5.1 is $(-,-)$. We thus obtain the following result:

Lemma 6.5.3. *Equation $(E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2})$ has a nonzero pair of rational solutions if and only if $A = B = \frac{1}{2}$ with set of steps $\mathcal{S}_3$. In this case, the solution is given by the pair $(\frac{1}{\widetilde{\gamma}_1}, -\frac{1}{\widetilde{\gamma}_2})$.*

<table>
<tr><th></th><th>$\mathcal{S}_1$</th><th>$\mathcal{S}_2$</th><th>$\mathcal{S}_3$</th><th>$\mathcal{S}_4$</th><th>$\mathcal{S}_5$</th></tr>
<tr><td>$(A,B) = (0,0)$</td><td colspan="5">$(+,+)$</td></tr>
<tr><td>$(A,B) = (0,\frac{1}{2})$</td><td>$(-,+)$</td><td>$(\perp,4)$</td><td>$(+,-)$</td><td colspan="2">$(\perp,4)$</td></tr>
<tr><td>$(A,B) = (\frac{1}{2},0)$</td><td>$(-,+)$</td><td>$(\perp,2)$</td><td colspan="2">$(-,+)$</td><td>$(\perp,2)$</td></tr>
<tr><td>$(A,B) = (\frac{1}{2},\frac{1}{2})$</td><td colspan="2" rowspan="2">$(+,+)$</td><td>$(-,-)$</td><td colspan="2">$(\perp,4)$</td></tr>
<tr><td>$A + B = 1$ and $(A,B) \neq (\frac{1}{2},\frac{1}{2})$</td><td colspan="3">$(\perp,2)$</td></tr>
<tr><td>A generic†</td><td colspan="5">$(\perp,2)$</td></tr>
<tr><td>B generic†</td><td colspan="5">$(\perp,4)$</td></tr>
</table>

Table 6.5.1 – The table summarizing solutions to the homogeneous equation $(E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2})$. Note that all cases are handled, as A and B are never equal to 1. We refer to the proof below for details on the signed entries.

Proof of Table 6.5.1. First, the entries of type $(\perp, i)$ can be directly checked by looking at the tables in computed in Section 6.4. It remains to prove the "signed" entries. To do this, we will apply Lemmas 6.5.1 and 6.5.2 for some sets of steps and various values of A and B to exhibit the signed solutions to $(E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2})$.

We first tell how to build the first four lines, that correspond to the cases $(A, B) \in \{0, \frac{1}{2}\}^2$. These conditions on A and B correspond to the fact that each individual function $\widetilde{\gamma}_1$ (depending on A) and $\widetilde{\gamma}_2$ (depending on B) admits a *signed decoupling*. More precisely, assume that we restrict to some set of steps $\mathcal{S}_i$ with $1 \leq i \leq 5$, and some value of $(A, B) \in \{0, \frac{1}{2}\}^2$. If one writes

$$\widetilde{\gamma}_1 = h_{1,1} \cdot h_{1,2} \text{ with } h_{1,1}^{\iota_1} = \varepsilon_{1,1} h_{1,1} \text{ and } h_{1,2}^{\iota_2} = \varepsilon_{1,2} h_{1,2}$$

†. A generic means that $A \neq \{0, \frac{1}{2}\}$ and $A + B \neq 1$ and B generic means that $B \neq \{0, \frac{1}{2}\}$.

and

$$\widetilde{\gamma}_2 = h_{2,1} \cdot h_{2,2} \text{ with } h_{2,1}^{\iota_1} = \varepsilon_{2,1} h_{2,1} \text{ and } h_{2,2}^{\iota_2} = \varepsilon_{2,2} h_{2,2},$$

then one obtains the following signed solution to ($E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2}$) for models of set of steps $\mathcal{S}_i$ and (A, B) having the prescribed value:

$$(h_1, h_2) = \left(\frac{h_{2,1}}{h_{1,1}}, \frac{h_{1,2}}{h_{2,2}}\right) \text{ with } h_1^{\iota_1} = \varepsilon_{1,1}\varepsilon_{2,1} h_1 = \varepsilon_1 h_1 \text{ and } h_2^{\iota_2} = \varepsilon_{1,2}\varepsilon_{2,2} h_2 = \varepsilon_2 h_2.$$

We thus only give below the signed decouplings of $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ relatively to the given set of steps and weights A, B. They cover the first four lines of Table 6.5.1. We write below $x = x(s)$ and $y = y(s)$.

1. **Signed decouplings of $\widetilde{\gamma}_1$:**

   (a) **$A = 0$, any set of steps:** In this case, we have $\widetilde{\gamma}_1 = -td_{-1,1}\frac{1}{y} \in \mathbb{C}(y)$, hence

   $$\widetilde{\gamma}_1 = h_{1,2} \text{ with } h_{1,2}^{\iota_2} = h_{1,2}.$$

   (b) **$A = \frac{1}{2}$, sets of steps $\mathcal{S}_1$, $\mathcal{S}_3$ and $\mathcal{S}_4$ (equivalently: $d_{1,0} = 0$)**: In this case, it is easy to check that

   $$(x\widetilde{\gamma}_1)^2 = \tfrac{1}{4} - t^2 d_{1,-1}d_{-1,1} - t^2 d_{1,-1}d_{0,1}x - t^2 d_{1,-1}d_{1,1}x^2 \in \mathbb{C}[x].$$

   This polynomial is never a square in $\mathbb{C}[x]$:

   — When $d_{1,1} = 0$, it has degree 1 in the variable x because then $d_{0,1} \neq 0$.

   — Otherwise, it has degree 2 in the variable x, with discriminant Δ equal to

   $$\begin{aligned}\Delta &= t^4 d_{1,-1}^2 d_{0,1}^2 + (1 - 4t^2 d_{1,-1}d_{-1,1})t^2 d_{1,-1}d_{1,1} \\ &= d_{1,-1}d_{1,1}t^2 + (d_{1,-1}^2 d_{0,1}^2 - 4d_{1,-1}d_{-1,1})t^4.\end{aligned}$$

   The coefficient in t^2 is nonzero since $d_{1,-1}$ and $d_{1,1}$ are nonzero. Hence, since t is transcendental over the field $\mathbb{Q}(d_{i,j})$, we deduce that $\Delta \neq 0$.

   Therefore, $x\widetilde{\gamma}_1$ does not belong to $\mathbb{C}(x(s))$ while $(x\widetilde{\gamma}_1)^2$ does. Since $\mathbb{C}(s)/\mathbb{C}(x)$ is Galois with Galois group ι_1, these conditions translate into $(x\widetilde{\gamma}_1)^{\iota_1} \neq x\widetilde{\gamma}_1$ and $\left((x\widetilde{\gamma}_1)^2\right)^{\iota_1} = (x\widetilde{\gamma}_1)^2$, so $(x\widetilde{\gamma}_1)^{\iota_1} = -x\widetilde{\gamma}_1$, and

   $$\widetilde{\gamma}_1 = h_{1,1} \text{ with } h_{1,1}^{\iota_1} = -h_{1,1}.$$

2. **Signed decouplings of $\widetilde{\gamma}_2$:**

   (a) **$B = 0$, any set of steps**: In this case, we have $\widetilde{\gamma}_2 = -td_{-1,1}\frac{1}{x} \in \mathbb{C}(x)$, and thus

   $$\widetilde{\gamma}_2 = h_{2,1} \text{ with } h_{2,1}^{\iota_1} = h_{2,1}.$$

(b) $\boldsymbol{B = \frac{1}{2}}$**, set of steps** $\boldsymbol{\mathcal{S}_1}$: In this case, from (iii) of Lemma 6.5.1 with $\lambda = \frac{1}{2}$ we have

$$\mu \stackrel{\text{def}}{=} -(y\widetilde{\gamma}_2)u_\lambda = -(\tfrac{1}{2} - td_{-1,1}\tfrac{x}{y})u_\lambda = \tfrac{1}{4} - t^2 d_{1,-1}d_{-1,1} - \tfrac{1}{2}td_{0,1}y \in \mathbb{C}[y].$$

Moreover, from (ii) of Lemma 6.5.1 with $\lambda = \frac{1}{2}$, then

$$u_\lambda^2 = \tfrac{1}{4} - t^2 d_{1,-1}d_{-1,1} - t^2 d_{1,-1}d_{0,1}x \in \mathbb{C}[x].$$

This polynomial is not a square in $\mathbb{C}(x)$ because it has degree 1 in x, so $u_\lambda^{\iota_1} = -u_\lambda$. Reasoning as above, we deduce

$$\widetilde{\gamma}_2 = h_{2,1} \cdot h_{2,2} \stackrel{\text{def}}{=} \frac{1}{u_\lambda} \cdot \left(-\frac{\mu}{y}\right) \text{ with } h_{2,1}^{\iota_1} = -h_{2,1} \text{ and } h_{2,2}^{\iota_2} = h_{2,2}.$$

(c) $\boldsymbol{B = \frac{1}{2}}$**, set of steps** $\boldsymbol{\mathcal{S}_3}$: In this case, from (ii) of Lemma 6.5.2 we have

$$(y\widetilde{\gamma}_2)^2 = \tfrac{1}{4} - d_{1,-1}d_{-1,1}t^2 - d_{1,1}d_{-1,1}t^2y^2 \in \mathbb{C}[y].$$

This polynomial is not a square in $\mathbb{C}[y]$. Indeed, it has degree 2, and its discriminant Δ is equal to

$$\Delta = (1 - 4d_{1,-1}d_{-1,1}t^2)d_{1,1}d_{-1,1}t^2 = d_{1,1}d_{-1,1}t^2 + O(t^4).$$

As t is transcendental over the field of parameters $\mathbb{Q}(d_{i,j})$, then Δ is always nonzero since $d_{1,1}$ and $d_{-1,1}$ are nonzero. Therefore, $(y\widetilde{\gamma}_2)^{\iota_2} = -y\widetilde{\gamma}_2$, from which we deduce

$$\widetilde{\gamma}_2 = h_{2,2} \text{ with } h_{2,2}^{\iota_2} = -h_{2,2}.$$

There now remains to fill line 5 of Table 6.5.1, which corresponds to the case of $A + B = 1$ for sets of steps $\mathcal{S}_1$ and $\mathcal{S}_2$. In this case, we have from (ii) of Lemma 6.5.1 with $\lambda = A$ that

$$(x\widetilde{\gamma}_1)u_\lambda = \lambda(1-\lambda) - t^2d_{1,-1}d_{-1,1} - (\lambda t d_{1,0} + t^2 d_{1,-1}d_{0,1})x \in \mathbb{C}[x].$$

Moreover, from (iii) of Lemma 6.5.1 with $\lambda = A$, then

$$-(y\widetilde{\gamma}_2)u_\lambda = \lambda(1-\lambda) - t^2d_{1,-1}d_{-1,1} - ((1-\lambda)td_{0,1} + t^2d_{-1,1}d_{1,0})y \in \mathbb{C}[y].$$

Note that $u_\lambda \neq 0$, for $(x\widetilde{\gamma}_1)u_\lambda$ is a nonzero polynomial in $\mathbb{C}[x]$ (the constant coefficient is a nonzero polynomial in $\mathbb{F}[t]$, for t is transcendental over $\mathbb{F}$ and $d_{1,-1}d_{-1,1} \neq 0$), x transcendental over $\mathbb{C}$. Therefore, the pair

$$(h_1, h_2) \stackrel{\text{def}}{=} \left(\frac{x}{x\widetilde{\gamma}_1 u_\lambda}, -\frac{y}{y\widetilde{\gamma}_2 u_\lambda}\right) \text{ with } h_1^{\iota_1} = h_1 \text{ and } h_2^{\iota_2} = h_2$$

is a signed solution to $(E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2})$. □

6.5.2 One particular case: support $\mathcal{S}_1$, $B = \frac{1}{2}$ and $A \neq \frac{1}{2}$

In the previous subsection, we were able to give a uniform proof for determining which parameters and supports allow for nonzero solutions to ($E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2}$). For ($E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega}$), we were not able to find a uniform argument, for one edge case remains, that we thus treat aside in this section. The remaining cases (i.e. $\mathcal{S}_1$ with $B \neq \frac{1}{2}$ or $A = \frac{1}{2}$ or the other set of steps) are then treated in Section 6.5.3.

This edge case concerns models with set of steps $\mathcal{S}_1$ ($d_{1,1} = d_{1,0} = 0$) and Boltzmann weights satisfying $B = \frac{1}{2}$ and $A \neq \frac{1}{2}$. We then show that the generating function $Q(x,y)$ is non-D-algebraic in x or y.

Following Lemma 6.5.1, let

$$u \stackrel{\text{def}}{=} u_{1/2} = \tfrac{1}{2} - td_{1,-1}\tfrac{x}{y} = -(\tfrac{1}{2} - td_{0,1}y - td_{-1,1}\tfrac{y}{x}) = \frac{1}{2} - A + x\widetilde{\gamma}_1.$$

This function satisfies the following relations.

Lemma 6.5.4. *Write $x = x(s)$ and $y = y(s)$. The following identities hold:*

(i) $u^2 = \frac{1}{4} - t^2 d_{-1,1}d_{1,-1} - t^2 d_{1,-1}d_{0,1}x \in \mathbb{C}[x]$ *and* $u^{\iota_1} = -u$,

(ii) $-(y\widetilde{\gamma}_2)u = \frac{1}{4} - \frac{1}{2}td_{0,1}y - t^2 d_{-1,1}d_{1,-1} \in \mathbb{C}[y]$.

Proof. The algebraic identities of (i) and (ii) are a direct consequence of Lemma 6.5.1.

Moreover, $u^2 \in \mathbb{C}(x)$, while from (i) this polynomial is not a square in $\mathbb{C}(x)$ (indeed, it has degree 1 in x). Since $\mathbb{C}(s)/\mathbb{C}(x)$ is Galois with Galois group generated by ι_1, we have $(u^2)^{\iota_1} = u^2$ and $u^{\iota_1} \neq u$, hence $u^{\iota_1} = -u$. □

We investigate the solutions of ($E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega}$).

Lemma 6.5.5. *If (h_1, h_2) is a solution to ($E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega}$), then $(h_2)_\infty = k \cdot (\iota_2 P_4 + P_4) + p \cdot (0 + \infty)$ for some $p \geq 0$ and $k \in \{0, 1\}$.*

Proof. From Table 6.4.1 and Proposition 6.3.10, we observe that the only nonnegative entry of the matrix M_2 is $\delta(\iota_2 P_4, P_4) = 1$. Therefore, from Lemma 6.3.7, we see that if (h_1, h_2) is a pair of solutions to ($E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega}$), then the poles of h_2 must belong to $\{P_4, \iota_2 P_4, 0, \infty\}$.

We now bound the order of the pole P_4 of h_2. We show that if it has order greater than 1, then P_4 is a pole of h_1, and deduce a contradiction. For the first part, we use that h_2 is a solution to ($E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega}$):

$$\widetilde{\gamma}_1 h_1 + \widetilde{\gamma}_2 h_2 + \omega = 0.$$

We first note that P_4 is a zero of order 1 of $\widetilde{\gamma}_2$ (the zeros of $\widetilde{\gamma}_1$ are computed in the Maple worksheet). Now, assume that P_4 is a pole of h_2 of order greater than 1. As P_4 is a zero of order 1 of $\widetilde{\gamma}_2$, we deduce that P_4 is a pole of $\widetilde{\gamma}_2 h_2$. Then from ($E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega}$), we deduce that P_4 is a pole of $\widetilde{\gamma}_1 h_1$. As $P_4 \neq 0, \infty$, it is not a pole of $\widetilde{\gamma}_1$ (Proposition 6.1.3), thus P_4 is a pole of h_1.

Since P_4 is a pole of h_1, then by Lemma 6.3.7, there must exist $Q^+ \in \mathcal{L}^+$ such that $\sigma^n P_4 = Q^+$ for some $n \geq 0$. But since $B = \frac{1}{2}$, Table 6.4.1 implies that $\delta(\iota_2 P_4, P_4) = \delta(\sigma^{-1} P_4, P_4) = 1$. Therefore, we deduce that

$$\delta(\iota_2 P_4, Q^+) = \delta(\iota_2 P_4, P_4) + \delta(P_4, Q^+) = 1 + n \geq 1.$$

This is a contradiction, since no entry of the line corresponding to $\iota_2 P_4$ in Table 6.4.1 is positive.

Therefore, the pole P_4 has order 0 or 1. Since h_2 satisfies $h_2^{\iota_2} = h_2$, the point $\iota_2 P_4$ has the same order as P_4 as a pole of h_2, and Table 6.4.1 and the fact that $B = \frac{1}{2}$ asserts that $P_4 \neq \iota_2 P_4$, hence the result. □

Lemma 6.5.6. *The function $u\widetilde{\gamma}_2$ has divisor $(u\widetilde{\gamma}_2) = P_4 + \iota_2 P_4 - 0 - \infty$.*

Proof. See the dedicated section in the Maple worksheet covering the set of steps $\mathcal{S}_1$. □

We can now state the classification for this support and parameters.

Proposition 6.5.7. *For every weighted model of the set of steps $\mathcal{S}_1$, if $B = \frac{1}{2}$ and $A \neq \frac{1}{2}$, then the series $Q(x,y)$ is non-D-algebraic in x and y.*

Proof. Assume that $Q(x,y)$ is x-D-algebraic or y-D-algebraic. By Proposition 6.1.5, Theorem 6.2.1 and Lemma 6.2.2, then $\widetilde{F}(s) = \widetilde{f}(s) + \widetilde{f}_h(s)$, $\widetilde{G}(s) = \widetilde{g}(s) + \widetilde{g}_h(s)$, with $(\widetilde{f}, \widetilde{g})$ solution to $(E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega})$ and $(\widetilde{f}_h, \widetilde{g}_h)$ solution to $(E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2})$. As $(E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2})$ has no nonzero rational solution by Lemma 6.5.3, this implies that $\widetilde{f}_h = \widetilde{g}_h = 0$, thus $\widetilde{F}(s) = \widetilde{f}(s)$ and $\widetilde{G}(s) = \widetilde{g}(s)$. We now distinguish between two cases, depending on the value of k in Lemma 6.5.5 ($k = 0$ or $k = 1$).

— If $k = 0$, then $(\widetilde{g})_\infty = p \cdot (0 + \infty)$. Thus, by Lemma 6.2.4, we have that $\widetilde{g}(s) = H(1/y(s))$ for $H(y) \in \mathbb{F}[y]$ a polynomial. Thus, Proposition 6.1.6 implies that $yQ(0,y) = H(1/y)$, which is absurd since $Q(0,y) = 1 + O(y)$.

— Otherwise, $k = 1$, and $(\widetilde{g})_\infty = P_4 + \iota_2 P_4 + p \cdot (0 + \infty)$. We thus have from Lemma 6.5.6

$$(\widetilde{g} \cdot (u\widetilde{\gamma}_2)) = (\widetilde{g})_0 + P_4 + \iota_2 P_4 - P_4 - \iota_2 P_4 - (p+1) \cdot (0 + \infty) = (\widetilde{g})_0 - (p+1) \cdot (0 + \infty).$$

Hence, the poles of $\widetilde{g} \cdot (u\widetilde{\gamma}_2)$ belong to $\{0, \infty\}$. Furthermore, we have from (ii) of Lemma 6.5.4 that $-4u\widetilde{\gamma}_2 = \frac{1-2td_{0,1}y(s)-4t^2d_{-1,1}d_{1,-1}}{y(s)} \in \mathbb{C}(y(s))$, thus $\widetilde{g} \cdot (u\widetilde{\gamma}_2)$ is fixed by ι_2. Thus, Lemma 6.2.4 implies that $\widetilde{g} \cdot (-4u\widetilde{\gamma}_2) = H(1/y(s))$ for some polynomial $H(y) \in \mathbb{F}[y]$, and thus

$$\widetilde{g}(s) = \frac{y(s)}{1 - 2td_{0,1}y(s) - 4t^2 d_{-1,1}d_{1,-1}} H(1/y(s))$$

for some polynomial $H(y) \in \mathbb{F}[y]$. Thus, Proposition 6.1.6 implies that

$$yQ(0,y) = \frac{y}{1 - 2td_{0,1}y - 4t^2 d_{-1,1}d_{1,-1}} H(1/y).$$

Since

$$yQ(0,y) = y + O(y^2)$$

and

$$\frac{y}{1-2td_{0,1}y-4t^2d_{-1,1}d_{1,-1}}H(1/y) = \frac{H(0)}{1-4t^2d_{-1,1}d_{1,-1}} + O(1/y),$$

this implies that $H(y) = \mu$ a constant in $\mathbb{F}$, and thus that $\widetilde{g}(s) = \mu/u$ for some $\mu \in \mathbb{F}$.
Therefore, we may rewrite ($E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega}$) as

$$\widetilde{\gamma}_1\widetilde{f} - \tfrac{\mu}{u} + \omega = 0. \tag{6.5.1}$$

But $u^{\iota_1} = -u$ by (i) of Lemma 6.5.4, thus $(x\widetilde{\gamma}_1)^{\iota_1} = (A - \frac{1}{2} + u)^{\iota_1} = A - \frac{1}{2} - u$, thus $(\widetilde{\gamma}_1)^{\iota_1} + (\widetilde{\gamma}_1) = (2A-1)/x$. Moreover, $\omega = 1 - A - B = \frac{1}{2} - A$ since $B = \frac{1}{2}$. Thus, by taking ι_1(6.5.1) + (6.5.1), one obtains the identity

$$(2A-1)\,\tfrac{\widetilde{f}}{x} - (2A-1) = 0.$$

As $A \neq \frac{1}{2}$, this implies that $\widetilde{f}(s) = x(s)$. By Proposition 6.1.6, this implies that $Q(x,0) = 1$, which is absurd since $a, b > 0$. □

6.5.3 Full classification

We now state and prove the full classification:

Theorem 6.5.8. *For any weighted genus 0 model, the generating function $Q(x,y)$ of weighted walks in the quadrant with interacting boundaries has the following nature in the variables x and y:*

1. *For all models of set of steps $\mathcal{S}_1$ or $\mathcal{S}_2$, and if $a+b = ab$, the generating function $Q(x,y)$ is **rational** with partial series $Q(x,0)$ and $Q(0,y)$ respectively equal to*

$$Q(x,0) = \frac{1}{1 - x\frac{ad_{1,0}t + abd_{1,-1}d_{0,1}t^2}{1-abd_{1,-1}d_{-1,1}t^2}}, \quad Q(0,y) = \frac{1}{1 - y\frac{bd_{0,1}t + abd_{-1,1}d_{1,0}t^2}{1-abd_{1,-1}d_{-1,1}t^2}}.$$

2. *For all models of set of steps $\mathcal{S}_3$ where $a = b = 2$, the generating function $Q(x,y)$ is **algebraic** of degree 4, with partial series $Q(x,0)$ and $Q(0,y)$ respectively equal to*

$$Q(x,0) = \frac{1}{\sqrt{1 - x^2\frac{4d_{1,1}d_{1,-1}t^2}{1-4d_{1,-1}d_{-1,1}t^2}}}, \quad Q(0,y) = \frac{1}{\sqrt{1 - y^2\frac{4d_{1,1}d_{-1,1}t^2}{1-4d_{1,-1}d_{-1,1}t^2}}}.$$

3. *In all other cases, the series $Q(x,y)$ is **neither x-D-algebraic nor y-D-algebraic**.*

Proof. We prove all the points in order. We begin with 1. Recall that as $A = 1 - \frac{1}{a}$ and $B = 1 - \frac{1}{b}$, we have $a + b = ab$ is equivalent to $A + B = 1$, which implies that $\omega = 0$. From (ii) and (iii) of Lemma 6.5.1 with $\lambda = A$, we see that the pair $\left(\frac{x(s)}{x(s)\widetilde{\gamma}_1 u_A}, -\frac{y(s)}{y(s)\widetilde{\gamma}_2 u_A}\right) \in \mathbb{C}(x(s)) \times \mathbb{C}(y(s))$ is a nonzero solution to $(E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega})$. Therefore, by (1) of Lemma 6.2.7 and (i) and (ii) of Lemma 6.5.1, there exists $\lambda \in \mathbb{C}$ such that

$$Q(x,0) = \frac{\lambda}{AB - t^2 d_{1,-1}d_{0,1}x - t^2 d_{1,-1}d_{-1,1} - Atd_{1,0}x}$$
$$\text{and } Q(0,y) = \frac{\lambda}{AB - t^2 d_{1,-1}d_{1,0}y - t^2 d_{1,-1}d_{-1,1} - Btd_{0,1}y}.$$

We know that $Q(0,0) = 1$, hence by substituting $x = 0$ in $Q(x,0)$ (or $y = 0$ in $Q(0,y)$), we find $\lambda = AB - t^2 d_{1,-1}d_{-1,1}$. We obtain the identities claimed in the theorem using $\frac{1}{A} = b$ and $\frac{1}{B} = a$ (this uses $a + b = ab$).

We now prove 2. In this case, we have from Lemma 6.5.3 that $(E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2})$ admits a nonzero solution $\left(\frac{1}{\widetilde{\gamma}_1}, -\frac{1}{\widetilde{\gamma}_2}\right)$. Therefore, by (2) of Lemma 6.2.7 and Lemma 6.5.2, there exists $\lambda \in \mathbb{C}$ such that

$$Q(x,0) = \frac{\lambda}{\sqrt{\frac{1}{4} - d_{1,-1}d_{-1,1}t^2 - d_{1,1}d_{-1,1}x^2t^2}} \quad \text{and } Q(0,y) = \frac{\lambda}{\sqrt{\frac{1}{4} - d_{1,-1}d_{-1,1}t^2 - d_{1,1}d_{-1,1}y^2t^2}}.$$

We know that $Q(0,0) = 1$, thus $\lambda = \sqrt{\frac{1}{4} - d_{1,-1}d_{-1,1}t^2}$, and we get the expression in the theorem.

Now, it remains to prove (3), namely that for all other cases $Q(x,y)$ is non-D-algebraic in x and y. Depending on the case, we use one of the three arguments below.

(1) If we are in the situation of Section 6.5.2, then $Q(x,y)$ is non-D-algebraic in x and y by Proposition 6.5.7.

(2) If the entries of M_1 satisfy $(M_1)_{i,j} \in \mathbb{Z}^- \cup \{\perp\}$, then by Lemma 6.3.11, if (h_1,h_2) is a rational solution to $(E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega})$, the poles of h_1 must belong to $\{0,\infty\}$. Moreover, Lemma 6.5.3 asserts that there is no nonzero rational solution to $(E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2})$ (the only situation where it happens corresponds to point 2 of the present theorem, and has already been covered). Therefore, by Lemma 6.2.5, the generating function $Q(x,y)$ is non-D-algebraic in x and y.

(2') Likewise, if the entries of M_2 satisfy $(M_2)_{i,j} \in \mathbb{Z}^- \cup \{\perp\}$, then for (h_1,h_2) a solution to $(E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega})$ the poles of h_2 must belong to $\{0,\infty\}$ by Lemma 6.3.11. Moreover, Lemma 6.5.3 asserts that there is no nonzero rational solution to $(E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2})$. Therefore, by Lemma 6.2.5, the generating function $Q(x,y)$ is non-D-algebraic in x and y.

We check that for any value of the parameters, we are in one of the three above cases (see Section 6.4).

	Support 1	Support 2	Support 3	Support 4	Support 5
(1) Proposition 6.5.7	$A + B \neq 1 \wedge B = \frac{1}{2}$				
(2) $(M_1)_{i,j} \in \mathbb{Z}^- \cup \{\bot\}$		$A + B \neq 1$	$A \neq \frac{1}{2}$		
(2′) $(M_2)_{i,j} \in \mathbb{Z}^- \cup \{\bot\}$	$A + B \neq 1 \wedge B \neq \frac{1}{2}$		$B \neq \frac{1}{2}$	always	
Algebraic solution	$A + B = 1$		$A = B = \frac{1}{2}$		

The above table gives the exact argument for each value of the parameters, and one can check that no case is missing. □

Chapitre 7

Quelques perspectives

Dans ce chapitre, on revient sur les résultats de la thèse, et réfléchit à leur éventuel prolongement. Je tiens à préciser que cette liste est non-exhaustive.

7.1 Orbite et approche formelle

7.1.1 Rendre systématique l'élimination de pôles

Dans la classification des modèles à petits pas à poids, le critère d'algébricité est précisément la conjonction des conditions (1) le groupe est fini et (2) la fraction xy admet un découplage rationnel (résultat résumé dans [DER24]). De manière similaire, pour un modèle à grands pas à petits pas arrière, si (1) l'orbite a été déterminée comme étant finie et qu'une paire d'invariants rationnels non triviaux en a donc été extraite, et (2) qu'un découplage rationnel de xy existe et a été calculé, on dispose de deux paires d'invariants, qu'il s'agit de combiner afin d'appliquer le lemme des invariants (Lemme 2.2.7) et conclure à l'algébricité. Dans tous les cas qui ont été traités dans [BH24], cette procédure a pu être menée à bien. Il est alors raisonnable de penser que ce critère s'étend aux modèles à grands pas à petits pas arrière, c'est à dire que lorsque l'orbite de $\mathcal{S}$ est finie, et que la fraction $x^{i+1}y^{j+1}$ admet un découplage rationnel, alors la série génératrice des chemins basés sur le modèle $\mathcal{S}$ partant du point (i, j) est algébrique.

La preuve du critère pour les modèles à petits pas à poids repose sur l'uniformisation de la courbe du noyau $\overline{E_t}$, de genre 0 ou 1, chose qu'il n'est pas possible de faire dans le cas général. Une stratégie de preuve générale basée sur la stratégie [BBR21] serait de prouver qu'une fois réunis les ingrédients (1) et (2), la procédure d'élimination des pôles conduisant à une paire d'invariants satisfaisant les conditions du lemme des invariants aboutit toujours, et dans ce cas de proposer un algorithme. On a déjà facilement le lemme 2.2.10 qui rend facile la vérification du caractère trivial des invariants dès lors que l'on parvient à éliminer les pôles en x et y des composantes des invariants. Il faudrait alors démontrer que les pôles provenant du découplage de la fraction relative au point de départ, et ceux provenant des invariants rationnels, permettent de produire une telle paire d'invariants. Enfin, il faudrait démontrer que les équations catalytiques ainsi formées sont toujours bien fondées, de manière à pouvoir appliquer [BJ06].

7.1.2 D'autres exploitations de la structure de l'orbite

Dans la littérature des marches dans le quart de plan, on appelle *sections* de l'équation les fonctions inconnues, qui sont des spécialisations de la série $Q(x,y,t)$ et de ses dérivées, dépendant d'au plus une variable catalytique qui apparaissent dans la partie droite de l'équation fonctionnelle. Par exemple, dans l'équation (1.2.3) qui apparait dans le cas de modèles à petits pas arrière, les sections sont les séries $Q(x,0,t)$, $Q(0,y,t)$ et $Q(0,0,t)$. Lorsque les pas qui reculent n'ont plus cette contrainte, on peut voir apparaître plus généralement les sections $\left(\partial_x^n \partial_y^m Q(x,y,t)\right)(z=0)$ pour tout m et n, et z étant soit x soit y. Chaque section introduit une nouvelle fonction inconnue, ce qui complexifie l'étude de l'équation. Aussi, on cherche à les éliminer.

Pour ce faire, les auteurs de [BBM21] évaluent l'équation sur des paires de fonctions (u,v) de l'orbite, de sorte que le noyau de l'équation vérifie $K(u,v,t) = K(x,y,t)$. Il faut noter que ce travail concerne essentiellement le cas d'orbites finies. En choisissant une certaine combinaison $\alpha_{(u,v)} \in \mathbb{Q}(u_1,\dots,u_m,v_1,\dots,v_n,t)$ (u_i et v_i étant les coordonnées qui apparaissent dans l'orbite) de ces équations (soit une somme d'orbite), les auteurs se ramènent dans bien des cas à une équation fonctionnelle de la forme

$$K(x,y,t)\left(\sum_{(u,v)\in\mathcal{O}} \alpha_{u,v} Q(u,v,t)\right) = H(u_1,\dots,u_m,v_1,\dots,v_n,t) \tag{7.1.1}$$

avec H une fraction rationnelle connue explicitement. Les auteurs conjecturent (Conjecture 4.2 [BBM21]) qu'une telle combinaison linéaire $\left(\alpha_{(u,v)}\right)_{(u,v)}$ existe toujours pour les modèles bidimensionels, et d'autre part qu'elle est unique si tous les pas qui avancent sont petits.

Les outils galoisiens développés dans [BH24] donnent de nouvelles pistes pour traiter cette question, dans le sens où elle est similaire à la question du découplage, déjà traitée dans cet article avec succès au moyen de sommes d'orbite données par la structure galoisienne de cette dernière. En outre, il existe une théorie des faisceaux sur des graphes, qui permettrait de mieux relier les propriétés d'élimination des sections avec les propriétés du graphe de l'orbite. On se basera notamment sur le développement élémentaire de l'article [Fri11].

7.2 Walks with interacting boundaries

In Theorem 6.5.8, we showed how the addition of the Boltzmann weights affects the nature of the generating function $Q(x,y)$ of walks with interacting boundaries for weighted models of genus 0. Namely, for the first two sets of steps, the relation $a+b = ab$ between the weights makes the series $Q(x,y)$ rational ; for the third set of steps the relation $a = b = 2$ makes the series $Q(x,y)$ algebraic ; while other Boltzmann weights and other sets of steps keep the series non x-D-finite nor y-D-finite. We now give some perspectives based on these results.

7.2.1 Phase transitions

Regarding the sets of steps $\mathcal{S}_1$ and $\mathcal{S}_2$, one may note that since there is an infinite number of Boltzmann weights a, b such that $Q(x,y)$ is explicit, the question of phase transitions introduced in [TOR14] can be partially treated on the curve $a + b = ab$ (a hyperbola).

Recall that the phases are defined as follows. Let $\mathcal{S}$ be a weighted model. For $n \geq 0$, denote $\mathbb{P}_n$ the probability on the walks using n steps defined by

$$\mathbb{P}_n(w) = \frac{\left(\prod_{(i,j)\in\mathcal{S}} d_{i,j}^{n_{i,j}}\right) a^{n_x} b^{n_y}}{[t^n]Q(1,1)}$$

(i.e. the probability of a walk using n steps is proportional to the numerator in the above equation). Define

$$\mathcal{A} \stackrel{\text{def}}{=} \limsup_n \mathbb{P}_n(\{w \text{ walk of } n \text{ steps } : \ w \text{ terminates on the } x\text{-axis}\})$$

and

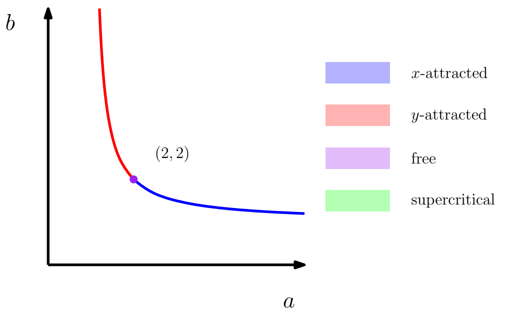
$$\mathcal{B} \stackrel{\text{def}}{=} \limsup_n \mathbb{P}_n(\{w \text{ walk of } n \text{ steps } : \ w \text{ terminates on the } y\text{-axis}\}).$$

These limits correspond respectively to $\mathcal{A}$ and $\mathcal{C}$ in [TOR14]). Four phases are then defined as follows :

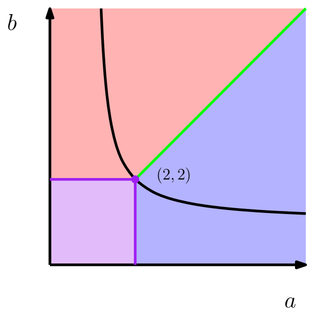
1. if $\mathcal{A} = \mathcal{B} = 0$, then the phase is *free* (the walk moves away from the axes),
2. if $\mathcal{A} > 0$ and $\mathcal{B} = 0$, then the phase is *x-attracted* (the walk moves away from the y-axis, and tends to come back infinitely often on the x-axis),
3. if $\mathcal{A} = 0$ and $\mathcal{B} > 0$, then the phase is *y-attracted* (the walk moves away from the x-axis, and tends to come back infinitely often on the y-axis),
4. if $\mathcal{A} > 0$ and $\mathcal{B} > 0$, then the phase is *supercritical* (the walk tends to come in contact with the axes infinitely often).

The values $\mathcal{A}$ and $\mathcal{B}$ can be expressed using the generating function $Q(x,y)$ as

$$\mathcal{A} = \limsup_n \frac{[t^n]Q(1,0)}{[t^n]Q(1,1)} \qquad \mathcal{B} = \limsup_n \frac{[t^n]Q(0,1)}{[t^n]Q(1,1)}.$$



(a) Phase diagram on $a + b = ab$.



(b) Conjectured phase diagram.

FIGURE 7.2.1 – Phase diagrams for model $\mathcal{S}_2$ with $d_{-1,1} = d_{1,-1} = d_{1,0} = d_{0,1} = 1$.

In the case of $\mathcal{S}_1$ or $\mathcal{S}_2$, the function $Q(x, y)$ is rational, hence the singularity analysis on the poles is straightforward. For instance, for the model $\mathcal{S}_2$ with $d_{i,j} = 1$, it yields the phase diagram in Figure 7.2.1a below (see the joint Maple worksheet).

The change of nature of the generating function $Q(x, y)$ on this curve, which contains the critical point (a_0, b_0) at the junction of the four phases suggests that this curve could be related to the phase transitions of the walk. Numerical computations allow us to conjecture that the full phase diagram looks like Figure 7.2.1b.

It would be interesting to know whether the knowledge of the phases on the curve $a + b = ab$ is enough to deduce some parts of the phase diagram 7.2.1b, mainly the part under the curve.

7.2.2 Combinatorial interpretation

For models $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$, we found algebraic solutions for $Q(x, y)$ when the weights are subject to some relations (i.e. $a + b = ab$ for $\mathcal{S}_1$ and $\mathcal{S}_2$; $a = b = 2$ for $\mathcal{S}_3$). These relations were found indirectly through the study of the q-difference equation. These relations being simple enough , one may wonder if the expressions found in (1) and (2) of Theorem 6.5.8 for those weights may be deduced through a more combinatorial argument.

Regarding the weights $a = b = 2$ for $\mathcal{S}_3$, Andrew Elvey-Price pointed out in a private communication with the author a direct proof through an adaptation of the reflection principle. The generating function of unconstrained two dimensional walks using the set of steps $\mathcal{S}_3$ that terminate on the x-axis is easily found to be

$$F(x, t) = \frac{1}{\sqrt{1 - 4xtd_{1,-1}\left(xtd_{1,1} + \frac{1}{x}td_{-1,1}\right)}}.$$

The reflection principle allows us to relate walks with interacting boundaries with Boltzmann weights $a = b = 2$ to these walks, through the following identity

$$Q(x, 0) \cdot \frac{1}{\sqrt{1 - 4t^2 d_{1,-1} d_{-1,1}}} = F(x, t),$$

which allows us to deduce the form of $Q(x,0)$.

Such a direct proof is yet to be found regarding the weights $a+b=ab$, and would be enlightening. For instance, it could give an alternative explanation as why the relation $a+b=ab$ changes the nature of $Q(x,y)$, and hopefully permit to find other sets of steps for which this relation between the Boltzmann weights yield a D-algebraic generating function.

7.2.3 Other q-difference equations

The general study of rational solutions to q-difference equations has already been investigated before. Depending on different constraints on the coefficients and the relation of the complex number q with regards to these coefficients, there may or may not be a general algorithm to decide whether such solutions exist.

In the general case where the coefficients of the equation and q may share algebraic relations, the problem is undecidable [Abr10].

In a more specific case, when the coefficients depend on one parameter, [AR13] gives an algorithm to determine numerical values of this parameter so that the q-difference equation has a nontrivial rational solution. Since we work with more parameters, and we want to find all the algebraic relations between them so that the equation has solutions, none of these algorithms can be applied verbatim. This justifies the approach taken in this chapter.

The author thinks that the approach taken in Section 6.3, specifically the structure given by Lemma 6.3.7, might adapt quite easily to the study of other decoupling equations of mixed type (multiplicative and additive). Moreover, we note that our approach works for a general infinite group of the walk, even if $\iota_2\iota_1$ is not presented as a multiplication by q on $\mathbb{P}^1$. More precisely, for a general decoupling equation of the form

$$uh_1 + vh_2 + w = 0$$

for functions u, v, w, $h_1^{\iota_1} = \pm h_1$ and $h_2^{\iota_2} = \pm h_2$ on some curve $\mathcal{C} \subset \mathbb{P}^1 \times \mathbb{P}^1$, the same technique yields similar finite sets $\mathcal{L}_1^-$, $\mathcal{L}_1^+$, $\mathcal{L}_2^-$ and $\mathcal{L}_2^+$, which may be exploited in the same way as in Section 6.3.

7.2.4 Extension to models of genus 1

For the time being, we only performed the systematic classification for walks with interacting boundaries of small steps of genus zero. It would seem natural to extend the methods for the models with small steps of genus 1 of [HS08; DR19; KR12] for the same purpose. Distinguishing between the finite group and infinite group cases, this could be the object of two upcoming papers.

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