WALKS IN THE QUADRANT WITH INTERACTING BOUNDARIES: GENUS ZERO CASE

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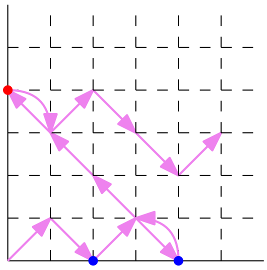
Abstract. The study of *lattice walks* restricted to the first quadrant has shed a lot of interest in the past twenty years. In particular, there has been an important effort to classify models of *weighted walks* with small steps with respect to the algebraic-differential nature of their generating function. The techniques that were developed in the course of this work are now applied to different extensions of those walks. One of these extensions, called *walks with interacting boundaries*, consists in accounting for the number of contacts of the walk with the axes, with motivation coming from statistical physics. These contacts are encoded as two additional parameters for the generating function, the *Boltzmann weights*.

For one notable family of models, called *genus zero models*, we establish in this paper the complete classification of their generating function, for all real values of the parameters. We do this by adapting to this more general case a method due to Dreyfus, Hardouin, Roques and Singer, used in the former classification, and which consists in studying the rational solutions to a *q-difference equation*. In almost all cases, we show that the generating function is hypertranscendental, regardless of the values of the weights. In the remaining cases, we prove that specific algebraic relations between the Boltzmann weights make the generating function ℕ-algebraic or ℕ-rational, contrasting with the interaction-less case.

Introduction

Lattice walks restricted to a cone are ubiquitous. Aside from their own interest (they naturally model random processes), these objects are general enough to represent many classes of discrete objects, including trees, permutations, planar maps, queuing processes. . .



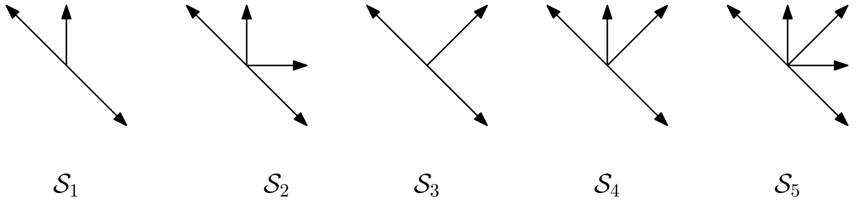
A walk in the quadrant

One important type of lattice walks are the two dimensional walks restricted to the first quadrant. In this setting, a *set of steps* $\mathcal{S}$ is set, and the goal is to enumerate the walks that use the steps in $\mathcal{S}$, and whose coordinates (i, j) are nonnegative at all times. The combination of two constraints ($i \geq 0$ and $j \geq 0$) make the nature of the problem extremely dependent on the set of steps $\mathcal{S}$, and highly nontrivial.

As a support for this claim, we mention the remarkable work of the classification of two dimensional quadrant walks with small steps (i.e. with $\mathcal{S} \subset \{-1,0,1\}^2$), whose goal was to determine the algebraic-differential nature of the generating function $Q(x,y,t)$ of such quadrant walks, counted with respect to ending coordinates and length. This systematic exploration showed a wide range of different behaviours depending on the set of steps. This achievement involved many results from enumerative combinatorics [BM10; Bou16], probability theory [DW15; FIM99], computer algebra [BBR21] and differential Galois theory [DHRS18; DHRS21], whose collective contributions gave rise to many effective tools to study *catalytic variables* equations. A reference sketching the combination of these different results to get the classification can be found for instance in [DER24]. The systematic study of quadrant walks continues to this day with other types of walks, for instance walks with arbitrarily large steps [BBM21; BH24], three-quadrant walks [Bou23], or the focus of the current paper, *walks with interacting boundaries*.

This extension, first introduced in [TOR14], consists in accounting for the number of contacts of the walk with the axes $i = 0$ and $j = 0$ (the *interaction*). The study of such problems leads to the study of the *phase transitions* of the model [Ren15]. More precisely, one parametrizes the tendency of the model to stick to the boundary $i = 0$ (resp. $j = 0$) via the *Boltzmann weight* $b \in \mathbb{R}^+$ (resp. $a \in \mathbb{R}^+$). The qualitative behaviour of the system changes depending on the Boltzmann weights, thus defining its phases. Counting the walks relative to the interaction statistics is done through functional equations that generalize those that appeared earlier, when ignoring this statistics. As a result, some of the techniques developed for counting walks in the quadrant may be applied to this setting (see [DR19]). For small steps models, they allowed authors to classify the generating function of some models for some Boltzmann weights [BOX21; BOR19], or even to solve them to get the full phase transition diagram such as in [TOR14].

Contribution. In this paper, we focus on the so-called *weighted models of genus zero*, which correspond to the five sets of steps below. For these sets of steps, we give the full classification of the algebraic-differential nature of the generating function $Q(x,y)$ of weighted walks with interacting boundaries with respect to the variables x and y. Recall that for every weighting $(d_v)_{v\in\mathcal{S}}$ attached to the steps and Boltzmann weights a and b, the generating function $Q(x,y)$ is defined as



The five considered sets of steps

$$Q(x,y) = \sum_{w \text{ walk}} \mathrm{weight}(w) x^i y^j t^n,$$

where for w a walk, (i,j) are its ending coordinates, n is its length, and $\mathrm{weight}(w)$ is a monomial depending on the weights $(d_v)_{v\in\mathcal{S}}$, a and b (defined in Section 1.1).

To do this, we adapt the strategy of [DHRS20] that was used to classify the generating function of weighted walks based upon on the sets of steps of Figure 1.1. This amounts to study the rational solutions of some functional equation (a *q-difference equation*), whose coefficients depend on the parameters a, b and $(d_v)_{v\in\mathcal{S}}$. This q-difference equation for the case of walks with interacting boundaries will be obtained in Section 1.3.

In Section 2 we exploit the symmetries of the q-difference equation to reduce the classification of its solutions to the study of two decoupling problems. We develop in Section 3 criteria based on pole propagation to test the existence of rational solutions to decoupling equations of a special form. These criteria reduce the classification of $Q(x,y)$ to relations between the weights, and they provide a concise way to explore the space of parameters, to either prove the nonexistence of such decouplings or to find solutions. These methods call for eventual generalizations (Section 6.3).

Using this technique, we ultimately obtain the form of the following theorem:

Theorem (Theorem 5.8, Section 5.3)**.** *For any weighted genus* 0 *model, the generating function* $Q(x,y)$ *of weighted walks in the quadrant with interacting boundaries has the following nature in the variables* x *and* y*:*

1. *For the sets of steps* $\mathcal{S}_1$ *or* $\mathcal{S}_2$ *and Boltzmann weights satisfying* $a+b=ab$*, the generating function* $Q(x,y)$ *is* ***rational*** *with specializations* $Q(x,0)$ *and* $Q(0,y)$ *respectively equal to*
$$Q(x,0)=\frac{1}{1-x\dfrac{ad_{1,0}t+abd_{1,-1}d_{0,1}t^2}{1-abd_{1,-1}d_{-1,1}t^2}},\qquad Q(0,y)=\frac{1}{1-y\dfrac{bd_{0,1}t+abd_{-1,1}d_{1,0}t^2}{1-abd_{1,-1}d_{-1,1}t^2}}.$$
2. *For the set of steps* $\mathcal{S}_3$ *and Boltzmann weights* $a=b=2$*, the generating function* $Q(x,y)$ *is* ***algebraic*** *of degree at most* 4*, with specializations* $Q(x,0)$ *and* $Q(0,y)$ *respectively equal to*
$$Q(x,0)=\frac{1}{\sqrt{1-x^2\dfrac{4d_{1,1}d_{1,-1}t^2}{1-4d_{1,-1}d_{-1,1}t^2}}},\qquad Q(0,y)=\frac{1}{\sqrt{1-y^2\dfrac{4d_{1,1}d_{-1,1}t^2}{1-4d_{1,-1}d_{-1,1}t^2}}}.$$
3. *In every other case, the series* $Q(x,y)$ *is* ***non x-D-algebraic nor y-D-algebraic*** *(meaning* $Q(x,y)$ *satisfies no polynomial differential equation in* x *nor in* y *for any choice of* x*,* y*,* t *and weighting* $(d_v)_{v\in\mathcal{S}}$*,* a *and* b*).*

In [DHRS20] where the Boltzmann weights a and b are both equal to one, the generating functions of the models were found to be all non x-D-algebraic nor y-D-algebraic. The addition of the Boltzmann weights a and b allows us to find algebraic models.

Organization of the paper. In Section 1, we recall standard definitions and facts in the study of quadrant walks, mainly their statistics, the weighting associated to a walk given a weighting $((d_v)_{v\in\mathcal{S}},a,b)$, the generating function $Q(x,y)$ of such walks, and the algebraic-differential classification of bivariate power series. We then focus on the five sets of steps of Figure 1.1, for which the kernel curve has genus 0. As a result, the kernel curve admits a rational parametrization $(x(s),y(s))$, for which we will recall basic facts, among which the existence of an automorphism $\sigma(s)\overset{\text{def}}{=}qs$ for some real number q which is not a root of unit. We then proceed to evaluate the functional equation for $Q(x,y)$ on this curve, this way obtaining two independent functional equations (q-difference equations) on the functions $\widetilde{F}(s)=Q(x(s),0)$ and $\widetilde{G}(s)=Q(0,y(s))$. We then compare the algebraic-differential properties of these two functions with those of $Q(x,y)$, thus reducing to the study of $\widetilde{F}(s)$ and $\widetilde{G}(s)$ through these q-difference equations.

In Section 2, we thus devise the strategy for determining the algebraic-differential nature of $\widetilde{F}(s)$ and $\widetilde{G}(s)$. The analytic properties of q-difference equations being rigid enough, a theorem due to Ishizaki allows us to reduce the classification to two *decoupling problems*, introduced in Lemma 2.2, one said homogeneous, the other one inhomogeneous. The classification will then go as follows. For most set of steps and weightings (see Section 5.2 for the one exception), and depending on the existence of solutions to these decoupling equations, either we will be in the case of Lemma 2.5, and then the generating function will not be D-algebraic in x and y, either we will be in the case of Lemma 2.7, and then we will be able to give explicit algebraic solutions for $Q(x,y)$.

Section 3 is thus devoted to the study of the rational solutions to decoupling equations of the form $\gamma_1(x(s),y(s))f(x(s))+\gamma_2(x(s),y(s))g(y(s))+c=0$ for all $s\in\mathbb{P}^1$, with fractions γ_1, γ_2 and c depending on the weights $(d_v)_{v\in\mathcal{S}}$, a, b and t. Through a process called *pole propagation*, we will see that the existence of rational solutions is conditioned to the relative positions of some particular points of $\mathbb{P}^1$ with respect to the action of σ. More explicitly, the relative position between two points is defined as the unique integer $n=\delta(P,Q)$ such that $\sigma^n P=Q$, which we call the *σ-distance*. In the end, we extract necessary conditions for the existence of rational solutions to the decoupling problem, based on the values $\delta(P,Q)$ for $(P,Q)\in\mathcal{L}^-\times\mathcal{L}^+$, for some finite sets $\mathcal{L}^-$ and $\mathcal{L}^+$.

Section 4 gives a way to compute this σ-distance, based on the fact that we can define valuations on the coordinates of the points that are considered in this paper (they are the orbits of points in $\mathcal{L}^-$ and $\mathcal{L}^+$). These valuations evolve with the action of σ in a deterministic way, allowing to effectively compute the σ-distance between two points given a fixed weighting. We extend this algorithm to find algebraic relations between the weights that guarantee a certain σ-distance. In the end, we compile the computations for our case in Appendix A.

In Section 5, we finally exploit the σ-distance computation of Section 4 along with the criteria determined in Section 3 to treat all the cases. In the end, we obtain the classification in the form of Theorem 5.8.

The very last Section 6 discusses various questions left at the end of the present paper, related to alternative proofs for the algebraic cases, the phase transitions of the models, and the general study of these inhomogeneous decoupling equations.

1. QUADRANT WALKS AND q-DIFFERENCE EQUATIONS

1.1. **Quadrant walks with interacting boundaries.**

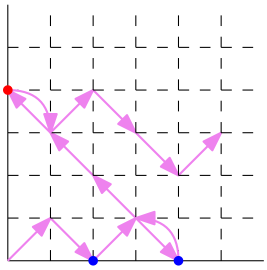


FIGURE 1.1. A walk in the quadrant with set of steps $\mathcal{S}=\{(-1,1),(1,-1),(1,1)\}$, using 13 steps, 2 contacts with the x-axis and 1 contact with the y-axis.

Quadrant walks. We recall here some basic definitions relative to the enumeration of walks in the quadrant. Consider a finite subset $\mathcal{S}$ of vectors in $\mathbb{Z}^2 \smallsetminus \{(0,0)\}$. A walk of n steps modeled on $\mathcal{S}$ (starting at $(0,0)$) is a sequence $w = v_1, v_2, \dots, v_n$ of steps $v_i \in \mathcal{S}$. We have the additional condition that the walk is a *quadrant walk*, that is, at every index $i \leq n$, we require that both coordinates of $\sum_{j\leq i} v_i$ are nonnegative. Figure 1.1 represents an example of a quadrant walk.

When counting these walks, several statistics are included. We list below the statistics of a walk w of n steps that we consider in this paper:

- the coordinates (i,j) of the last point of $\mathbb{N}^2$ they visit.
- for each step v in $\mathcal{S}$ the number n_v of occurrences of this particular step in the walk w.
- the number of contacts (also called interactions) of w with the axes. Recall that a contact with the x-axis occurs at each $i \geq 1$ such that $\sum_{j\leq i} v_j$ is zero. Thus, performing twice the step $(1,0)$ starting from $(0,0)$ accounts for two contacts with the x-axis, despite the walk remaining on the x-axis. The number of contacts with the x-axis (resp. y-axis) is denoted by n_x (resp. n_y). The *interacting boundaries* qualification refers to this statistics.

Weighted walks. In combinatorics, it is quite common to associate weights to objects, depending on their statistics, often for probabilistic purposes. Below, we define the weight of a quadrant walk with interacting boundaries.

For each $v \in \mathcal{S}$ is given $d_v > 0$ the real positive weight associated with the step v. Moreover, we are also given positive real numbers $a > 0$ (resp. $b > 0$) the *Boltzmann weight* associated with the x-axis (resp. y-axis). The weight of the walk w is then defined as the monomial

$$\operatorname{weight}(w) := \left(\prod_{v\in\mathcal{S}} d_v^{n_v}\right) a^{n_x} b^{n_y}.$$

This definition of weight is correlated with an associated probability distribution on walks of length n, defined so that the probability of a walk w is proportional to its weight, i.e.

$$\mathbb{P}_n(w) := \frac{\operatorname{weight}(w)}{\sum\limits_{w' \text{ walk of length } n} \operatorname{weight}(w')}.$$

The weighting induces a bias on the distribution of quadrant walks, for instance a bigger value for d_v increases the probability of performing the step v, or a small value of b favors walks that have fewer contacts with the y-axis. The above heuristics can be made precised through limit properties of the probability $\mathbb{P}_n$, that define *phases* (see Section 6.1).

We call a *weighted model* of quadrant walks with interacting boundaries a set of steps $\mathcal{S}$ together with a *weighting*, i.e. positive real weights $(d_v)_{v\in\mathcal{S}}$, and Boltzmann weights $a > 0$ and $b > 0$. Given a weighted model, we write

$$\mathbb{F} \overset{\text{def}}{=} \mathbb{Q}((d_v)_{v\in\mathcal{S}}, a, b)$$

for the subfield of $\mathbb{C}$ generated by the weights.

Generating function and functional equation. Given a weighted model $(\mathcal{S}, (d_v)_{v\in\mathcal{S}}, a, b)$, the generating function of quadrant walks on this weighted model is defined as

$$Q(x,y) \overset{\text{def}}{=} \sum_{w \text{ walk}} \operatorname{weight}(w) x^i y^j t^n = \sum_{w \text{ walk}} \left(\prod_{v\in\mathcal{S}} d_v^{n_v}\right) a^{n_x} b^{n_y} x^i y^j t^n.$$

Since there is a finite number of quadrant walks of length n and that the walks always terminate in the first quadrant, the generating function $Q(x,y)$ belongs to $\mathbb{F}[x,y][[t]]$ (recall that $R[[t]]$ denotes the ring of formal power series in the variable t with coefficients in the ring R). Note that it is harmless to have the d_v, a and b as real numbers for the exact counting of walks with regards to the statistics n_v, n_x and n_y. As the transcendence degree of $\mathbb{R}$ over $\mathbb{Q}$ is infinite, one may choose algebraically independent weights d_v, a and b over $\mathbb{Q}$, and still perform coefficient extraction to get these statistics since the coefficient $[t^n]Q(x,y)$ belongs to $\mathbb{Q}[(d_v)_{v\in\mathcal{S}},a,b,x,y]$. This is why we directly consider the generating function of weighted walks, only having x, y and t as variables.

The generating function is characterized through a functional equation. In Theorem 6 of [BOR19] the authors derive the following explicit functional equation for the series $Q(x,y)$ when the steps of the model $\mathcal{S}$ are small (that is $\mathcal{S}\subseteq\{-1,0,1\}^2$):

$$\begin{aligned} K(x,y)Q(x,y) &= \frac{xy}{ab} + x\left(y - \frac{y}{a} - tA_{-1}(x)\right)Q(x,0) \\ &\quad + y\left(x - \frac{x}{b} - tB_{-1}(y)\right)Q(0,y) - \left(\frac{xy}{ab}(1-a)(1-b) - t\varepsilon\right)Q(0,0). \end{aligned} \tag{1.1}$$

This functional equation generalizes those found in the quadrant walks literature for the study of weighted models independently of the interaction statistics. The polynomial $K(x,y) \overset{\text{def}}{=} xy(1-tS(x,y))$ is commonly called the *kernel*, where $S(x,y) \overset{\text{def}}{=} \sum_{(i,j)\in\mathcal{S}} d_{i,j}x^iy^j$ encodes the set of steps as a Laurent polynomial. The fractions $A_i(x)$ and $B_i(y)$ are then defined as $A_i(x) \overset{\text{def}}{=} [y^i]S(x,y)$ and $B_j(y) \overset{\text{def}}{=} [x^j]S(x,y)$. Finally, according to the notation of [BOR19], the variable ε is set to 1 if the step $(-1,-1)$ is an element of $\mathcal{S}$, and $\varepsilon = 0$ otherwise.

This functional equation is part of the class of polynomial equations in two *catalytic variables*, x and y, as the equation relates $Q(x,y)$ with its specializations $Q(x,0)$, $Q(0,y)$ and $Q(0,0)$. While the theory behind polynomial equations involving only one catalytic variable is well known (the solutions are always algebraic, see [BJ06]), this is not at all the case for equations of two catalytic variables. The systematic study of walks in the quadrant revolves around such equations, which is one of the reasons why this topic is challenging.

The differential classification. In general, we do not expect to find a closed form for $Q(x,y)$ from this equation. Thus, a more reasonable question is the *classification* of the generating function, that is knowing where the function $Q(x,y)$ fits in the following *algebraic-differential hierarchy*:

$$\text{rational} \subset \text{algebraic} \subset \text{D-finite} \subset \text{D-algebraic}.$$

We recall that the power series $Q(x,y)$ is called *rational* if it belongs to $\mathbb{K}(x,y,t)$; *algebraic* if it is a solution to a polynomial equation with coefficients in $\mathbb{K}(x,y,t)$; *x-D-finite* (resp. y-D-finite, t-D-finite) if it is a solution to a linear differential equation with respect to the variable x (resp. y, t) with coefficients in $\mathbb{K}(x,y,t)$, and it is *D-finite* if it is all at once x, y and t-D-finite; *x-D-algebraic* (resp. y-D-algebraic, t-D-algebraic) if it is a solution to a polynomial differential equation with respect to the variable x (resp. y, t) with coefficients in $\mathbb{K}(x,y,t)$, and it is *D-algebraic* if it is all at once x, y and t-D-algebraic.

The problem is to classify the models of quadrant walks according to where their generating function $Q(x,y)$ occurs in the hierarchy, and we say that a weighted model is of class X if its generating function is of class X. Such a classification gives qualitative information on the complexity of the walk. If it is low (at most D-finite), one can expect a nicer combinatorial interpretation, maybe a closed form, and fast algorithms to compute the coefficients. Otherwise, their study is more complicated, and requires specific techniques for each equation.

When setting the weights d_v, a and b to 1 (which amounts to ignoring the interaction and weights statistics, and is the original setting of the systematic classification), the classification of small steps models was completed in 2018. The methods of this first classification extend when considering arbitrary positive real weights d_v, and it is now complete as well (see [DR19]).

Some of these techniques may in turn be adapted to the study of walks with the interacting boundaries statistics, where we allow other values of a and b, and the weighted models of walks with interacting boundaries that have been studied up to now rely on the finiteness of the *group of the walk*, which is a group of birational transformations attached to each weighted model. This is the case in [TOR14], where the walks with interacting boundaries are completely solved for one specific model (the reversed Gessel model, also called Gouyou-Beauchamps) for all weights $a, b > 0$, establishing the full phase diagram. This is also the case in [BOX21], where the authors fully solve the Kreweras and reverse Kreweras walks with interaction for any value of the Boltzmann weights. Finally, in [BOR19], the authors systematically investigate the models having a finite group, for some Boltzmann weights, mainly (a,a), $(1,b)$, $(a,1)$ and (a,b) for a and b algebraically independent over $\mathbb{Q}$, giving upper bounds on the complexity of the generating function $Q(1,1)$. We propose to treat a case with an infinite group, and for nongeneric weights $d_{i,j}$, a and b.

1.2. **Genus zero models.** The authors of [DHRS21] define five models called the *genus zero models* (the terminology is explained in the next paragraph). They are listed in Figure 1.2 below, and they will be referenced in this paper as $\mathcal{S}_1$, $\mathcal{S}_2$, etc. The goal of the current paper is to establish the full classification of walks with interacting boundaries based on these sets of steps.

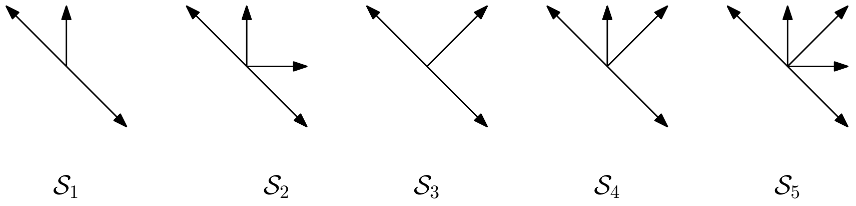


FIGURE 1.2. The five models of genus 0

For these sets of steps, we may perform simplifications on the general functional equation (1.1). First, the Laurent polynomial is of the form

$$S(x,y) = d_{1,-1}\frac{x}{y} + d_{-1,1}\frac{y}{x} + d_{1,0}x + d_{0,1}y + d_{1,1}xy,$$

with $d_{1,-1}$ and $d_{-1,1}$ always nonzero, and at least one of the $d_{0,1}$, $d_{1,0}$ or $d_{1,1}$ nonzero. Moreover, we have that $A_{-1}(x) = d_{1,-1}x$ and $B_{-1}(y) = d_{-1,1}y$, and since no model contains the step $(-1,-1)$, the variable ε is always zero. Finally, for every model of Figure 1.2, the power series $Q(0,0)$ is equal to 1. Indeed, any nontrivial walk one of these models has an ending point (i,j) satisfying $i + j > 0$ by an easy induction. Hence setting both x and y to 0 in $Q(x,y)$ leaves only the term in t^0, which is equal to 1. Summarizing these simplifications, the functional equation (1.1) can be rewritten as

$$\begin{aligned}K(x,y)Q(x,y) &= \frac{xy}{ab}(a+b-ab)\\ &\quad + x\left(y - \frac{y}{a} - td_{1,-1}x\right)Q(x,0) + y\left(x - \frac{x}{b} - td_{-1,1}y\right)Q(0,y).\end{aligned}$$

We now introduce the following notations:

$$\begin{aligned}&A \overset{\text{def}}{=} 1 - \frac{1}{a}, && B \overset{\text{def}}{=} 1 - \frac{1}{b}, && \omega \overset{\text{def}}{=} \frac{1}{ab}(a+b-ab),\\ &\gamma_1(x,y) \overset{\text{def}}{=} \frac{A}{x} - \frac{td_{1,-1}}{y}, && \gamma_2(x,y) \overset{\text{def}}{=} \frac{B}{y} - \frac{td_{-1,1}}{x}, && \gamma(x,y) \overset{\text{def}}{=} \frac{\gamma_1(x,y)}{\gamma_2(x,y)}.\end{aligned} \tag{1.2}$$

With these notations, the functional equation can finally be rewritten as

$$K(x,y)Q(x,y) = \omega xy + x^2y\gamma_1(x,y)Q(x,0) + xy^2\gamma_2(x,y)Q(0,y). \tag{1.3}$$

In the remaining of the section, we show how to further exploit the particularity of the genus 0 models to study (1.3), and in the end classify $Q(x,y)$ for any weighting on one of these set of steps.

The kernel curve. Given a weighted model, consider the kernel polynomial $K(x,y,t)$, which belong to $\mathbb{Q}(d_{i,j})[x,y,t]$. For every complex number t, this polynomial defines a complex affine curve in $\mathbb{C}^2$ defined as

$$E_t \overset{\text{def}}{=} \{(x,y) \in \mathbb{C}\times\mathbb{C} : K(x,y) = 0\}.$$

One considers the projective completion of this curve in $\mathbb{P}^1 \times \mathbb{P}^1$. Here, $\mathbb{P}^1 = \mathbb{P}^1(\mathbb{C})$ designates the complex projective line, which may be defined as the quotient of $\mathbb{C}\times\mathbb{C}$ by the equivalence relation

$$(x_0,x_1) \sim (x_0',x_1') \iff \exists\lambda \in \mathbb{C}^\star, (\lambda x_0, \lambda x_1) = (x_0',x_1').$$

One denotes by $[x_0 : x_1]$ the class of (x_0,x_1). The set $\mathbb{C}$ embeds into $\mathbb{P}^1$ through the map $x \mapsto [x:1]$, and we furthermore denote $\infty := [1:0]$. The projective completion of E_t in $\mathbb{P}^1\times\mathbb{P}^1$ is then

$$\overline{E_t} \overset{\text{def}}{=} \{([x_0:x_1],[y_0:y_1]) \in \mathbb{P}^1\times\mathbb{P}^1 : x_1^2y_1^2K(\tfrac{x_0}{x_1},\tfrac{y_0}{y_1}) = 0\}.$$

In [DHRS21] the authors examine the $\overline{E_t}$ that occur for weighted models with small steps. They prove that apart from trivial cases, the curve $\overline{E_t}$ is irreducible. When it is irreducible, they show that it is either nonsingular of genus 1 or of genus 0 with a unique singular point $\Omega = ([0:1],[0:1])$. By abuse of notation, a weighted model is said to be of genus g if the curve $\overline{E_t}$ has genus g.

It turns out that to study the models of genus 0, it is enough to consider the five fundamental sets of steps of Figure 1.2, hence the qualification of genus zero models.

Group and parametrization of the kernel curve. When the kernel curve has genus 0, the authors of [DHRS21] construct a specific rational parametrization $\phi : \mathbb{P}^1 \to \overline{E_t}$. We summarize basic facts and vocabulary on this parametrization, following Section 4.1 of [DHRS21].

We first recall basic facts about the *group of the walk*. It has been well established since the beginning of the study of quadrant walks in [FIM99] or [BM10] that for any model of walk with small steps, the projective curve $\overline{E_t}$ is equipped with two involutive automorphisms ι_1

and ι_2. They have the property that for all $(u,v) \in \mathbb{P}^1 \times \mathbb{P}^1$, $\iota_1(u,v) = (u,v')$ for some v' and $\iota_2(u,v) = (u',v)$ for some u'. For the models of genus zero, their expression simplifies as follows:

$$\iota_1([1:x_1],[1:y_1]) = \left([1:x_1],\left[1:\frac{d_{-1,1}x_1^2+d_{0,1}x_1+d_{1,1}}{d_{1,-1}y_1}\right]\right), \tag{1.4}$$

$$\iota_2([1:x_1],[1:y_1]) = \left(\left[1:\frac{d_{1,-1}y_1^2+d_{1,0}y_1+d_{1,1}}{d_{-1,1}x_1}\right],[1:y_1]\right). \tag{1.5}$$

Note that we choose to write them for points of $\mathbb{P}^1 \times \mathbb{P}^1$ written in homogeneous coordinates $([1:x_1],[1:y_1])$ for reasons detailed in Section 4. The group of the walk is then defined as the group of automorphisms of $\overline{E_t}$ generated by ι_1 and ι_2. These two involutions induce an automorphism σ of $\overline{E_t}$ defined as

$$\sigma \overset{\text{def}}{=} \iota_2 \circ \iota_1.$$

As a side note, if τ is an automorphism of $\mathbb{P}^1$ and $h(s) \in \mathbb{C}(s)$, then we will write $h^{\tau}(s) \overset{\text{def}}{=} h(\tau(s))$. The reason for the exponential notation is because the composition action of automorphisms of $\mathbb{P}^1$ on the function field of $\mathbb{P}^1$ is a right action, so that $h^{\tau_1\tau_2} = (h^{\tau_1})^{\tau_2}$.

We are now going summarize the properties of the parametrization ϕ of $\overline{E_t}$ which was defined in [DHRS20], and that we will use in the present paper. It is constructed so that the action of the group of the walk lifts through ϕ in a nice way.

We recall basic facts on *divisors* of a function field of an algebraic curve. We will be even more specific, and restrict to $\mathbb{P}^1$, whose function field is $\mathbb{C}(s)$ with s transcendental. A comprehensive introduction to these notions is contained in [Sti09].

Definition/Proposition 1.1 (Chapter 1 of [Sti09])**.** A *divisor* is a formal finite sum of points $D = \sum_{P\in\mathbb{P}^1} n_P P$ where n_P are integers. The *degree* of a divisor $D = \sum_P n_P P$ of $\mathbb{P}^1$ is defined as $\deg D = \sum_P n_P$. The map $D \mapsto \deg D$ is a group homomorphism. The following properties hold:

(1) Let h be a nonzero function in $\mathbb{C}(s)$. The function h has finitely many zeros in $\mathbb{P}^1$. The *zero divisor* of h is thus defined as

$$(h)_0 = \sum_{P \text{ zero of } h} \operatorname{ord}_P(h) \cdot P$$

where $\operatorname{ord}_P(h)$ is the multiplicity of P as a zero of h. Similarly, the *polar divisor* $(h)_\infty$ of h is defined as the zero divisor of h^{-1}.

(2) The *principal divisor* associated to a nonzero function h is defined as

$$(h) = (h)_0 - (h)_\infty.$$

It has the property that $(h) = 0$ if and only if $h \in \mathbb{C}$.

(3) For u, v two nonzero functions in $\mathbb{C}(s)$, then $(1/u) = -(u)$ and $(uv) = (u) + (v)$.

(4) For $h \notin \mathbb{C}$, the following holds:

$$1 \le \deg(h)_0 = \deg(h)_\infty = [\mathbb{C}(s) : \mathbb{C}(h(s))] < \infty$$

(recall that for A an extension, $[A:k]$ denotes the dimension of A as a k-vector space).

We may now state some properties of the parametrization $\mathbb{P}^1 \to \overline{E_t}$.

Proposition 1.2 (Section 4.1 of [DHRS21])**.** *For any model of Figure 1.2 and weighting d_v, and any real number $t \in (0,1)$ transcendental over $\mathbb{Q}(d_{i,j},a,b)$, the following assertions hold.*

(1) There exists a rational parametrization $\phi : s \mapsto (x(s),y(s))$ of $\mathbb{P}^1$ onto $\overline{E_t}$. The fractions $x(s)$ and $y(s)$ both belong to $\overline{\mathbb{F}(t)}(s)$, where $\overline{\mathbb{F}(t)}$ is the algebraic closure of $\mathbb{F}(t)$.

(2) *The parametrization ϕ is one-to-one everywhere except for $\phi(0) = \phi(\infty) = \Omega$, with $\Omega \overset{\text{def}}{=} ([0:1],[0:1])$, i.e. $(x(0), y(0)) = (x(\infty), y(\infty)) = \Omega$.*

(3) *The divisors (Proposition 1.1) of the functions $x \overset{\text{def}}{=} x(s)$ and $y \overset{\text{def}}{=} y(s)$ on the curve $\mathbb{P}^1$ are*

$$(x) = 0 + \infty - Q_1 - Q_2, \qquad (y) = 0 + \infty - Q_3 - Q_4,$$

for some points $Q_i \neq 0, \infty$ of $\mathbb{P}^1$ (that do not have to be distinct).

(4) *The group lifts in the following way. There exists a real number $q \notin \{-1, 1\}$, with q algebraic over $\mathbb{F}(t)$, such that for all $s \in \mathbb{P}^1$ one has*

$$\iota_1(\phi(s)) = \phi(\tfrac{1}{s}) \text{ and } \iota_2(\phi(s)) = \phi(\tfrac{q}{s}).$$

When the context is clear, we will also denote by ι_1, ι_2 and σ the automorphisms on $\mathbb{P}^1$ defined by

$$\iota_1(s) \overset{\text{def}}{=} \tfrac{1}{s} \qquad \iota_2(s) \overset{\text{def}}{=} \tfrac{q}{s} \qquad \sigma(s) \overset{\text{def}}{=} qs.$$

As the multiplicative order of q is infinite, so is the order of σ. Moreover, the only points of $\mathbb{P}^1$ whose orbit under the action of σ is finite are 0 and ∞.

Proposition 1.3 (Section 1.3 of [DHRS20])**.** *The function field $\mathbb{C}(s)/\mathbb{C}$ of $\mathbb{P}^1$ has the following lattice.*

$$\begin{tikzcd}
& \mathbb{C}(s) \arrow[dl, dash, "\iota_1"'] \arrow[dr, dash, "\iota_2"] & \\
\mathbb{C}(x(s)) \arrow[dr, dash] & & \mathbb{C}(y(s)) \arrow[dl, dash] \\
& \mathbb{C} &
\end{tikzcd}$$

(1) *The extension $\mathbb{C}(s)/\mathbb{C}(x(s))$ is Galois of degree 2, with Galois group generated by the involution ι_1. This means that if $h = h^{\iota_1}$, then $h \in \mathbb{C}(x(s))$.*

(2) *Similarly, the extension $\mathbb{C}(s)/\mathbb{C}(y(s))$ is Galois of degree 2, with Galois group generated by the involution ι_2.*

(3) *The field of constants $\mathbb{C}$ is the intersection $\mathbb{C}(x(s)) \cap \mathbb{C}(y(s))$, and as a result it is also the subfield of functions fixed by ι_1 and ι_2. Moreover, if $f(s)$ is fixed by σ, then $f(s)$ belongs to $\mathbb{C}$.*

1.3. **The q-difference equations.** We now fix the setting in which we can evaluate the functional equation (1.3) for $Q(x, y)$ on the curve $\overline{E_t}$ through the parametrization ϕ, in the manner of Section 2.1 of [DHRS21]. Once the conditions of this evaluation are fixed, this will transform the catalytic equation on $Q(x, y)$ (a power series in x and y) into a functional equation relating $Q(x(s), 0)$ and $Q(0, y(s))$ (meromorphic functions over $\mathbb{C} \subset \mathbb{P}^1$). The symmetries of $x(s)$ and $y(s)$ with respect to the group (Proposition 1.2) will then allow us to construct a functional equation on $Q(x(s), 0)$ only.

We first study the divisors of the functions

$$\widetilde{\gamma}_1(s) \overset{\text{def}}{=} \gamma_1(x(s),y(s)) \quad \widetilde{\gamma}_2(s) \overset{\text{def}}{=} \gamma_2(x(s),y(s)) \tag{1.6}$$

on $\mathbb{P}^1$ through a routine computation. Recall that

$$\gamma_1(x,y) = \frac{A}{x} - t\frac{d_{1,-1}}{y} \qquad\qquad \gamma_2(x,y) = \frac{B}{y} - t\frac{d_{-1,1}}{x},$$

as defined in (1.2).

Proposition 1.4. *For any real number t defined as in Proposition 1.2, the extensions $\mathbb{C}(s)/\mathbb{C}(\widetilde{\gamma}_1)$ and $\mathbb{C}(s)/\mathbb{C}(\widetilde{\gamma}_2)$ have degree two, and the functions $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ have the following divisors:*

$$(\widetilde{\gamma}_1) = P_1 + P_2 - 0 - \infty, \qquad\qquad (\widetilde{\gamma}_2) = P_3 + P_4 - 0 - \infty.$$

The points $P_1, \dots, P_4$ are distinct from 0 and ∞.

Proof. In this proof, we write $x = x(s)$ and $y = y(s)$, these functions thus satisfying $K(x,y) = 0$ for $K(X,Y)$ the kernel polynomial. We only perform the proof for $\widetilde{\gamma}_1 = \frac{A}{x} - \frac{td_{1,-1}}{y}$, the study of $\widetilde{\gamma}_2$ being symmetric. The computations can be followed in the Maple worksheet. We are going to prove that the minimal polynomial of x over $\mathbb{C}(\widetilde{\gamma}_1)$ has degree 2, thus showing that $[\mathbb{C}(x) : \mathbb{C}(\widetilde{\gamma}_1)] = 2$. We will then prove that $\widetilde{\gamma}_1$ has the announced poles 0 and ∞.

We first produce a vanishing polynomial of x over $\mathbb{C}(\widetilde{\gamma}_1)$. By definition, the polynomial $K(X,y) = 0$ is a vanishing polynomial of x over $\mathbb{C}(y)$. Hence, expressing y in terms of x and $\widetilde{\gamma}_1$ (which we can do since $td_{1,-1}$ is nonzero), we are left to consider the polynomial

$$\begin{aligned} P(X) = {} & (d_{1,1}d_{1,-1}t^2 - d_{1,0}t\widetilde{\gamma}_1 + \widetilde{\gamma}_1^2)X^2 \\ & + (d_{0,1}d_{1,-1}t^2 + Ad_{1,0}t + (1-2A)\widetilde{\gamma}_1)X \\ & + d_{1,-1}d_{-1,1}t^2 + A^2 - A, \end{aligned}$$

which by construction is a vanishing polynomial of x with coefficients in $\mathbb{C}[\widetilde{\gamma}_1] \subset \mathbb{C}(\widetilde{\gamma}_1)$. Moreover, $P(X)$ is nonzero, for its constant coefficient $d_{1,-1}d_{-1,1}t^2 + A^2 - A \in \mathbb{F}[t]$ is nonzero. Indeed, the real number t is transcendental over $\mathbb{F}$ and the coefficient $d_{1,-1}d_{-1,1}$ of t is nonzero for all genus zero models (Figure 1.2). Hence, x is algebraic over $\mathbb{C}(\widetilde{\gamma}_1)$.

We now show that $P(X)$ is irreducible in $\mathbb{C}(\widetilde{\gamma}_1)[X]$. First note that the leading coefficient $d_{1,1}d_{1,-1}t^2 - d_{1,0}t\widetilde{\gamma}_1 + \widetilde{\gamma}_1^2$ of $P(X)$ is nonzero. Indeed, it is a nonzero polynomial of $\mathbb{C}[\widetilde{\gamma}_1]$, with $\widetilde{\gamma}_1$ transcendental over $\mathbb{C}$ (the function x is both transcendental over $\mathbb{C}$ and algebraic over $\mathbb{C}(\widetilde{\gamma}_1)$). Hence, $P(X)$ is a degree two polynomial. In order to show that it is irreducible, we thus compute its discriminant $\Delta \in \mathbb{C}(\widetilde{\gamma}_1)$ and show that it cannot be a square in $\mathbb{C}(\widetilde{\gamma}_1)$. The discriminant is expressed as follows:

$$\begin{aligned} \Delta = {} & (-4d_{1,-1}d_{-1,1}t^2 + 1)\widetilde{\gamma}_1^2 \\ & + (4d_{1,0}d_{1,-1}d_{-1,1}t^3 - 4Ad_{0,1}d_{1,-1}t^2 + 2d_{0,1}d_{1,-1}t^2 - 2Ad_{1,0}t)\widetilde{\gamma}_1 \\ & + (d_{0,1}^2d_{1,-1}^2t^4 - 4d_{1,1}d_{1,-1}^2d_{-1,1}t^4 + 2Ad_{0,1}d_{1,0}d_{1,-1}t^3 \\ & \quad + A^2d_{1,0}^2t^2 - 4A^2d_{1,1}d_{1,-1}t^2 + 4Ad_{1,1}d_{1,-1}t^2). \end{aligned}$$

We see that Δ belongs to $\mathbb{C}[\widetilde{\gamma}_1]$, hence Δ is a square in $\mathbb{C}(\widetilde{\gamma}_1)$ if and only if Δ is a square in $\mathbb{C}[\widetilde{\gamma}_1]$. In turn, as Δ has degree two as a polynomial in $\widetilde{\gamma}_1$ (the leading coefficient $-4d_{1,-1}d_{-1,1}t^2 + 1$ is nonzero since t is transcendental over $\mathbb{F}$), Δ is a square if and only if its discriminant δ with respect to the variable $\widetilde{\gamma}_1$ is zero. The discriminant δ factors in $\mathbb{Q}[d_{i,j}, A][t]$ into $\delta = 16t^2 f_1 f_2 f_3$, the factors f_i written in the table below.

f_1	$d_{1,-1}$
f_2	$d_{1,-1}d_{-1,1}t^2 + A(A-1)$
f_3	$(d_{0,1}^2 d_{1,-1} + d_{1,0}^2 d_{-1,1} - 4d_{1,1}d_{1,-1}d_{-1,1})t^2 + d_{0,1}d_{1,0}t + d_{1,1}$

As t is transcendental over $\mathbb{F}$, the polynomial δ is zero if and only if one f_i is zero if and only if all coefficients of one f_i viewed as a polynomial in $\mathbb{F}[t]$ are zero. Define an ideal I of $\mathbb{Z}[d_{i,j}, A]$ by $I = d_{1,-1}d_{-1,1}(d_{1,1}, d_{0,1}, d_{1,0})$, and for $i \in \{1,2,3\}$ the ideal J_i of $\mathbb{Z}[d_{i,j}, A]$ generated by the coefficients in t of f_i. Then one sees that $I \subset J_1$, $I \subset J_2$, and finally through elimination that $I^3 \subset J_3$. Hence, if δ is zero, then $(d_{1,-1}, d_{-1,1}, d_{0,1}, d_{1,0}, d_{1,1}, A)$ must satisfy either $d_{1,-1} = 0$, or $d_{-1,1} = 0$, or $d_{1,0} = d_{0,1} = d_{1,1} = 0$, which is never the case given the constraints on the supports (Figure 1.2). Therefore, δ is always nonzero, Δ is never a square, and we conclude that $P(X)$ is always irreducible in $\mathbb{C}(\widetilde{\gamma}_1)[X]$. Thus, since P is an irreducible vanishing polynomial of x, and since $\mathbb{C}(x, \widetilde{\gamma}_1) = \mathbb{C}(x, y)$, this proves that $[\mathbb{C}(x,y) : \mathbb{C}(\widetilde{\gamma}_1)] = [\mathbb{C}(x, \widetilde{\gamma}_1) : \mathbb{C}(\widetilde{\gamma}_1)] = 2$.

We now conclude on the poles of $\widetilde{\gamma}_1$. By Proposition 1.2, the divisors of x and y are respectively $(x) = 0 + \infty - Q_1 - Q_2$ and $(y) = 0 + \infty - Q_3 - Q_4$ for some points Q_i. Hence, the poles of $\widetilde{\gamma}_1 = \frac{A}{x} - t\frac{d_{1,-1}}{y}$ are either 0 or ∞, which both have order at most 1. But by Proposition 1.1, $\deg(\widetilde{\gamma}_1)_\infty = [\mathbb{C}(s) : \mathbb{C}(\widetilde{\gamma}_1)] = 2$, thus we conclude that $\widetilde{\gamma}_1$ has these two poles, and thus $(\widetilde{\gamma}_1)_\infty = 0 + \infty$. □

We now determine for which real numbers t the composition of the functional equation (1.3) with $(x(s), y(s))$ (which both depend on t) is well defined.

Proposition 1.5. *There exists a positive real number $r > 0$ such that for every real number $t < r$ transcendental over $\mathbb{F}$, there exist two open sets U_0 and U_∞ of $\mathbb{P}^1$ such that $0 \in U_0$, $\infty \in U_\infty$, and so that the functions $Q(x(s), y(s))$, $Q(x(s), 0)$, $Q(0, y(s))$, are analytic on $U_0 \cup U_\infty$. Moreover, there exists an open set V satisfying $0 \in V \subset U_0$ such that $\iota_2(V) \subset U_\infty$ and $\sigma^{-1}(V) \subset U_0$.*

Proof. We are going to show that for t small enough, the series $Q(x,y) \in \mathbb{C}[x,y][[t]]$ is convergent on $\{(x,y) \in \mathbb{C} : |x|, |y| < 1\}$. Let $|x|, |y| < 1$, and $M \overset{\text{def}}{=} \sup\{|d_{i,j}|, |a|, |b|\}$ (note that M is positive). Then each walk w of length n contributes to a value of norm at most M^{2n} to the coefficient of t^n in $Q(x,y)$. Indeed, each performed step involves at most two weights (one $d_{i,j}$ and possibly one additional a or b). As the steps are small, the coefficient corresponding to w has norm at most $M^{2n}|x|^n|y|^n \leq M^{2n}$, for $|x|, |y| < 1$. Moreover, the set of steps of any of the considered weighted models is finite of cardinal at most 5 (Figure 1.2), hence the coefficient of t^n in $Q(x,y)$ has norm of at most $\left(5M^2\right)^n$. Hence, when $|x|, |y| < 1$, the power series $Q(x,y) \in \mathbb{C}(x,y)[[t]]$ has a positive radius of convergence $\rho \geq \frac{1}{5M^2}$. We thus fix a real number t so that $0 < t < \rho$ and t is transcendental over the field $\mathbb{F}$. Since the coefficients in t of $Q(x,y)$ are polynomials in x and y, the function $Q(x,y)$ is analytic in x and y at $(0,0)$.

We now study the convergence of the composition of the functions appearing in (1.3) with the parametrization $\phi(s) = (x(s), y(s))$. First, the functions $x(s)$ and $y(s)$ belong to $\mathbb{C}(s)$ with $x(0) = y(0) = x(\infty) = y(\infty) = 0$, thus they are both analytic at the points 0 and ∞. Thus, by composition, the functions $Q(x(s), y(s))$, $Q(x(s), 0)$ and $Q(0, y(s))$ are analytic at 0 and ∞. This proves the existence of the two announced open sets $U_0 \ni 0$ and $U_\infty \ni \infty$.

Finally, we construct V to be $U_0 \cap \iota_2^{-1}(U_\infty) \cap \sigma(U_0)$. This is open because $\iota_2(s) = \frac{q}{s}$ and $\sigma(s) = qs$, which are continuous functions in $\mathbb{P}^1 \to \mathbb{P}^1$. Moreover, V contains 0 because $\iota_2(\infty) = \iota_1(\infty) = \sigma(0) = \sigma^{-1}(0) = 0$. □

We now fix some small enough t transcendental over $\mathbb{F}$ prescribed by Proposition 1.5, and a parametrization ϕ accordingly. The evaluation of (1.3) on $(x(s), y(s))$ for s in $U_0 \cup U_\infty$ is thus well defined, yielding after dividing by $x(s)y(s)$ the following equation of meromorphic functions on $U_0 \cup U_\infty$:

$$0 = \omega + x(s)\widetilde{\gamma}_1(s)Q(x(s), 0) + y(s)\widetilde{\gamma}_2(s)Q(0, y(s)). \tag{1.7}$$

Define two analytic functions on the open set V by

$$\breve{F}(s) \overset{\text{def}}{=} x(s)Q(x(s), 0), \qquad \breve{G}(s) \overset{\text{def}}{=} y(s)Q(0, y(s)).$$

We now use (1.7) to construct meromorphic continuations $\widetilde{F}$ of $\breve{F}$ and $\widetilde{G}$ of $\breve{G}$ to the whole complex plane $\mathbb{C}$, by showing like in [DHRS20] that $\breve{F}$ satisfies a q-difference equation. We introduce the function

$$\widetilde{\gamma}(s) \overset{\text{def}}{=} \gamma(x(s), y(s)) = \frac{\widetilde{\gamma}_1(s)}{\widetilde{\gamma}_2(s)}, \tag{1.8}$$

where $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ were introduced in (1.6), and we rewrite (1.7) as follows for s in V.

$$-y(s)Q(0, y(s)) = \frac{\omega}{\widetilde{\gamma}_2(s)} + \widetilde{\gamma}(s)x(s)Q(x(s), 0). \tag{1.9}$$

By Proposition 1.5, if s is in V then $\frac{q}{s} = \iota_2(s) \in U_\infty$, hence from (1.7) we also have the following equation for s in V:

$$-y(\tfrac{q}{s})Q(0, y(\tfrac{q}{s})) = \frac{\omega}{\widetilde{\gamma}_2(\frac{q}{s})} + \widetilde{\gamma}(\tfrac{q}{s})x(\tfrac{q}{s})Q(x\left(\tfrac{q}{s}\right), 0). \tag{1.10}$$

We now use the symmetries of the functions $x(s)$ and $y(s)$. For all s in V, we have by Proposition 1.2 that $y(\frac{q}{s}) = y(s)$, hence $-y(\frac{q}{s})Q(0, y(\frac{q}{s})) = -y(s)Q(0, y(s))$. Moreover, we also have $x(\frac{q}{s}) = x(\frac{s}{q})$, hence $x(\frac{q}{s})Q(x(\frac{q}{s}), 0) = x(\frac{s}{q})Q(x(\frac{s}{q}), 0)$. Finally, as s is in V, the complex number $\frac{s}{q} = \sigma^{-1}(s)$ is also in U_0 by Proposition 1.5, so we can replace $x(\frac{q}{s})Q(x(\frac{q}{s}), 0)$ with $\breve{F}(\frac{s}{q})$ in (1.10). Hence, eliminating $y(s)Q(0, y(s))$ between (1.9) and (1.10) yields the following q-difference equation on $\breve{F}(s)$ for all s in V:

$$\breve{F}(\tfrac{s}{q}) = \frac{\widetilde{\gamma}}{\widetilde{\gamma}^{\iota_2}}(s)\breve{F}(s) + \left(\frac{\omega}{\widetilde{\gamma}_2(s)} - \frac{\omega}{\widetilde{\gamma}_2^{\iota_2}(s)}\right)\frac{1}{\widetilde{\gamma}^{\iota_2}(s)}.$$

As the absolute value of q is not equal to 1 (Proposition 1.2), this functional equation allows us to construct a unique continuation $\widetilde{F}$ of $\breve{F}$ meromorphic on the whole complex plane $\mathbb{C}$, satisfying the same equation:

$$\widetilde{F}(\tfrac{s}{q}) = \frac{\widetilde{\gamma}}{\widetilde{\gamma}^{\iota_2}}(s)\widetilde{F}(s) + \left(\frac{\omega}{\widetilde{\gamma}_2(s)} - \frac{\omega}{\widetilde{\gamma}_2^{\iota_2}(s)}\right)\frac{1}{\widetilde{\gamma}^{\iota_2}(s)}. \tag{1.11}$$

Indeed, assuming that $\widetilde{F}$ is a meromorphic continuation of $\breve{F}$ on some open set U, the functional equation (1.11) relates $\widetilde{F}(s)$ with $\widetilde{F}(\frac{s}{q})$ over $\mathbb{C}(s)$, which allows us to extend uniquely $\widetilde{F}$ as a meromorphic function on $qU \cup U \cup q^{-1}U$. As $|q| \neq 1$ (Proposition 1.2), we have that $\bigcup_{n \in \mathbb{Z}} q^n U = \mathbb{C}$, hence this process gives a unique meromorphic continuation $\widetilde{F}$ of $\breve{F}$ on $\mathbb{C}$.

Now, the functions $\breve{F}$ and $\breve{G}$ satisfy the linear relation (1.7) over $\mathbb{C}(s)$. This relation provides a unique meromorphic continuation $\widetilde{G}$ of $\breve{G}$ to $\mathbb{C}$ such that $\widetilde{F}$ and $\widetilde{G}$ satisfy

$$\widetilde{\gamma}_1(s)\widetilde{F}(s) + \widetilde{\gamma}_2(s)\widetilde{G}(s) + \omega = 0. \tag{1.12}$$

Finally, from (1.11) and (1.12), it is easy to see that the function $\widetilde{G}$ satisfies the following q-difference equation:

$$\widetilde{G}(qs) = \frac{\widetilde{\gamma}^{\iota_1}}{\widetilde{\gamma}}(s)\widetilde{G}(s) + \left(\frac{\omega}{\widetilde{\gamma}_1(s)} - \frac{\omega}{\widetilde{\gamma}_1^{\iota_1}(s)}\right)\widetilde{\gamma}^{\iota_1}(s). \tag{1.13}$$

1.4. **The D-algebraicity of $Q(x,y)$, $\widetilde{F}(s)$ and $\widetilde{G}(s)$.** The algebraic-differential properties of the formal power series $Q(x,y)$, $Q(x,0)$ and $Q(0,y)$ and their meromorphic counterparts $\widetilde{F}(s)$ and $\widetilde{G}(s)$ are related. The following proposition relates the x and y-D-algebraicity of $Q(x,y)$ over $\mathbb{C}(x,y)$ with the s-D-algebraicity of $\widetilde{F}(s)$ and $\widetilde{G}(s)$ over $\mathbb{C}(s)$. As t is a fixed real number, the study of the t-D-algebraicity of $Q(x,y,t)$ is not easily related to the properties of $\widetilde{F}(s)$ and $\widetilde{G}(s)$, and rigid parametrizations are needed (see [DH21]), which are not implemented in this paper.

Proposition 1.6. *For $t > 0$ as in Proposition 1.5, and a weighting $(d_v)_v$, a and b, the following statements are equivalent:*

(a) $Q(x,0)$ *is* x*-D-algebraic,*
(a') $Q(0,y)$ *is* y*-D-algebraic,*
(b) $\widetilde{F}(s)$ *is* s*-D-algebraic,*
(b') $\widetilde{G}(s)$ *is* s*-D-algebraic,*
(c) $Q(x,y)$ *is* x*-D-algebraic for all* y*,*
(c') $Q(x,y)$ *is* y*-D-algebraic for all* x*.*

Proof. We recall that on the open set V, we have

$$\widetilde{F}(s) = x(s)Q(x(s),0), \qquad \widetilde{G}(s) = y(s)Q(0,y(s)).$$

Up to a restriction of V, the maps $x(s)$ and $y(s)$ are biholomorphisms on V. The equivalence between (a) and (b) (resp. (a') and (b')) now follows from Lemmas 6.3 and 6.4 of [DHRS18].

Moreover, the functions $\widetilde{F}$ and $\widetilde{G}$ are linearly related over $\mathbb{C}(s)$ by Equation (1.12), which shows the equivalence between (b) and (b'), and thus the equivalence between (a), (a'), (b) and (b').

Finally, (c) is equivalent to (a) from Equation (1.3). Indeed, $\partial_x Q(0,y) = 0$, so $Q(0,y)$ is x-D-algebraic. Hence, since $x^2 y\gamma_1(x,y)$ and $K(x,y)$ are both nonzero elements of $\mathbb{C}(x,y)$, the closure properties of the x-D-algebraic class imply that $Q(x,y)$ is x-D-algebraic if and only if $Q(x,0)$ is. Similarly, (c') is equivalent to (a'). □

Therefore, determining the algebric-differential nature of the function $Q(x,y)$ is equivalent to determining the differential nature of either $\widetilde{F}(s)$ or $\widetilde{G}(s)$, which satisfy functional equations with more structure: q-difference equations. Equations (1.11) and (1.13) do not completely characterize $\widetilde{F}(s)$ and $\widetilde{G}(s)$, so we will use in a crucial way the fact that they continue the functions $x(s)Q(x(s),0)$ and $y(s)Q(0,y(s))$, on which we have some grasp through their power series expansion, that give information on the poles near 0 and ∞ of $\widetilde{F}$ and $\widetilde{G}$.

Finally, we will often make use of the following proposition, which allows us to go from meromorphic functions on $\mathbb{C}$ to power series, mostly to obtain equations on $Q(x,0)$ (resp. $Q(0,y)$) from equations on $\widetilde{F}(s)$ (resp. $\widetilde{G}(s)$).

Proposition 1.7. *Assume that a Laurent series $H(x) \in \mathbb{C}((x))$ induces a meromorphic function at $x = 0$. If $H(x(s)) = 0$ or if $H(y(s)) = 0$ for s near 0, then $H(x) = 0$.*

Proof. Let W be a neighborhood of 0 such that $H(x(s)) = 0$ for all s in $W \setminus \{0\}$. The function $x : \mathbb{C} \to \mathbb{C}$ is non-constant and analytic at 0 with $x(0) = 0$. Hence by the open mapping theorem for holomorphic functions, the image of W under x is an open neighborhood of 0. Thus, the analytic function $H(x)$ is locally zero at 0, hence zero by analytic continuation. The argument is similar for $H(y(s)) = 0$. ☐

2. Classification strategy

In the previous section, we have constructed two q-difference equations (1.11) and (1.13) satisfied by two meromorphic functions on $\mathbb{C}$, whose differential properties reflect those of the generating functions of quadrant walks $Q(x,y)$. There are many results on the differential transcendence of power series solution to a q-difference equation. One of the first of those results was proved by Ishizaki for the solutions of equations of the form $y(qs) = a(s)y(s) + b(s)$ [Ish98]. We apply it to equations (1.11) and (1.13):

Proposition 2.1. *The following statements are equivalent:*

1. *$\widetilde{F}$ and $\widetilde{G}$ are D-algebraic over $\mathbb{C}(s)$.*
2. *$\widetilde{F}$ and $\widetilde{G}$ are in $\mathbb{C}(s)$.*

Proof. The real number q is not a root of unit, the coefficients of (1.11) and (1.13) belong to $\mathbb{C}(s)$, and the functions $\widetilde{F}$ and $\widetilde{G}$ are meromorphic at $s = 0$. Hence we may apply Theorem 1.2 of [Ish98] to the two equations, which shows the claim. ☐

Thus, to investigate the D-algebraicity of $\widetilde{F}$ (or equivalently $\widetilde{G}$), the strategy that we are going to explain in this section will consist in determining for which weighted models these functions can be rational. In the earlier paper applying this strategy [DHRS20] (corresponding to the case without the interaction statistics $a = b = 1$), the authors prove that there exists no rational solution. Whether it exists is highly dependent on the coefficients of the q-difference equations considered. For the classification of the models with the five supports of Figure 1.2, we find a general strategy which allows us to handle almost all the cases uniformly, its culmination being the classification in Theorem 5.8.

Because of the symmetries of the coefficients of equations (1.11) and (1.13), we will reduce the study of rational solutions to these equations to what we call *decoupling equations*. Recall that the kernel curve $\overline{E_t}$ admits a rational parametrization $(x,y) : \mathbb{P}^1 \longrightarrow \overline{E_t}$. Consider some fraction $h(x,y) \in \mathbb{C}(x,y)$. The problem is to find two fractions $f(x)$ and $g(y)$ so that the following equation holds for all points s in $\mathbb{P}^1$,

$$h(x(s),y(s)) = f(x(s)) + g(y(s)).$$

This is called an *additive decoupling of h*, this notion being introduced in [BBR21]. Likewise, one can wonder if there exist $f(x)$ and $g(y)$ so that for all points s in $\mathbb{P}^1$,

$$h(x(s),y(s)) = f(x(s))g(y(s)).$$

This is called a *multiplicative decoupling of h* (introduced in [BPFHR25]).

The existence of such decouplings for a fraction $h(x,y)$ plays an important role in the classification of generating functions enumerating walks. For instance, in the case of quadrant walks with small steps without interacting boundaries, algebraicity is characterized by the finiteness of the group, and the fact that the fraction xy admits an additive decoupling (see [DER24]). In our case, the equations that appear are generalizations of the decoupling equations above, mixing the additive and multiplicative form, and they serve the same purpose: we will see that the fact that they admit a solution or not determines the position of $Q(x,y)$ in the differential hierarchy. This explains our choice of terminology, as we introduce them now.

Lemma 2.2. *Assume that the functions $\widetilde{F}(s)$ and $\widetilde{G}(s)$ (defined in Section 1.3) are rational. In this case, define the following elements of $\mathbb{C}(s)$:*

$$\widetilde{f}(s) \overset{\text{def}}{=} \frac{1}{2}\left(\widetilde{F}(s)+\widetilde{F}^{\iota_1}(s)\right), \qquad \widetilde{g}(s) \overset{\text{def}}{=} \frac{1}{2}\left(\widetilde{G}(s)+\widetilde{G}^{\iota_2}(s)\right),$$

$$\widetilde{f}_h(s) \overset{\text{def}}{=} \frac{1}{2}\left(\widetilde{F}(s)-\widetilde{F}^{\iota_1}(s)\right), \qquad \widetilde{g}_h(s) \overset{\text{def}}{=} \frac{1}{2}\left(\widetilde{G}(s)-\widetilde{G}^{\iota_2}(s)\right).$$

(1) The pair $(h_1(s),h_2(s)) = (\widetilde{f}(s),\widetilde{g}(s))$ satisfies the inhomogeneous equation

$$(E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega}) \qquad \widetilde{\gamma}_1(s)h_1(s)+\widetilde{\gamma}_2(s)h_2(s)+\omega = 0 \text{ with } h_1^{\iota_1} = h_1 \text{ and } h_2^{\iota_2} = h_2.$$

(2) The pair $(h_1(s),h_2(s)) = (\widetilde{f}_h(s),\widetilde{g}_h(s))$ satisfies the homogeneous equation

$$(E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2}) \qquad \widetilde{\gamma}_1(s)h_1(s)+\widetilde{\gamma}_2(s)h_2(s) = 0 \text{ with } h_1^{\iota_1} = -h_1 \text{ and } h_2^{\iota_2} = -h_2.$$

We refer to equations $(E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega})$ and $(E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2})$ as the decoupling equations*.*

Proof. Assuming that $\widetilde{F}$ and $\widetilde{G}$ belong to $\mathbb{C}(s)$, we first find a relation between $\widetilde{F}^{\iota_1}$ and $\widetilde{G}^{\iota_2}$ (composition with ι_1 and ι_2 is always well defined for rational functions). Recall that $\widetilde{F}(\frac{s}{q}) = \widetilde{F}^{\iota_1\iota_2}$, hence we may rewrite the q-difference equation (1.11) into

$$\left(\widetilde{\gamma}(s)\widetilde{F}^{\iota_1}(s)+\frac{\omega}{\widetilde{\gamma}_2(s)}\right)^{\iota_2} = \widetilde{\gamma}(s)\widetilde{F}(s)+\frac{\omega}{\widetilde{\gamma}_2(s)},$$

hence by applying ι_2 on both sides we obtain

$$\widetilde{\gamma}(s)\widetilde{F}^{\iota_1}(s)+\frac{\omega}{\widetilde{\gamma}_2(s)} = \left(\widetilde{\gamma}(s)\widetilde{F}(s)+\frac{\omega}{\widetilde{\gamma}_2(s)}\right)^{\iota_2}. \tag{2.1}$$

Moreover, the linear relation (1.12) between $\widetilde{F}$ and $\widetilde{G}$ can be rewritten as

$$-\widetilde{G}(s) = \widetilde{\gamma}(s)\widetilde{F}(s)+\frac{\omega}{\widetilde{\gamma}_2(s)},$$

so by applying ι_2 we obtain

$$-\widetilde{G}^{\iota_2}(s) = \left(\widetilde{\gamma}(s)\widetilde{F}(s)+\frac{\omega}{\widetilde{\gamma}_2(s)}\right)^{\iota_2}. \tag{2.2}$$

Eliminating the right-hand sides between (2.1) and (2.2), we extract the following relation between $\widetilde{F}^{\iota_1}$ and $\widetilde{G}^{\iota_2}$:

$$\widetilde{\gamma}_1(s)\widetilde{F}^{\iota_1}(s)+\widetilde{\gamma}_2(s)\widetilde{G}^{\iota_2}(s)+\omega = 0. \tag{2.3}$$

We copy for convenience Equation (1.12):

$$\widetilde{\gamma}_1(s)\widetilde{F}(s)+\widetilde{\gamma}_2(s)\widetilde{G}(s)+\omega = 0. \tag{2.4}$$

Taking the average of equations (2.3) and (2.4), one obtains $(E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega})$. Taking half the difference of equations (2.3) and (2.4), one obtains $(E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2})$. Indeed, ι_1 is an involution, hence $(\widetilde{F}^{\iota_1})^{\iota_1} = \widetilde{F}$, so $\widetilde{f}^{\iota_1} = \widetilde{f}$ and $\widetilde{f}_h^{\iota_1} = -\widetilde{f}_h$. The same argument applies to $\widetilde{g}$ and $\widetilde{g}_h$. □

Remark 2.3. From Proposition 1.3, the condition $\widetilde{f}^{\iota_1} = \widetilde{f}$ is equivalent to the condition that there exists $f(x) \in \mathbb{C}(x)$ such that $\widetilde{f}(s) = f(x(s))$. Likewise, the condition $\widetilde{f}^{\iota_1} = -\widetilde{f}$ asserts that there exists $\widetilde{f}(s)^2 = f(x(s))$ for some f, but that $\widetilde{f}(s)$ itself is not a function of $x(s)$. This explains the qualification of *decoupling equations*: they relate functions in two different variables. ■

We will see in the remaining of the section how the study of the rational solutions of the decoupling equations gives information on the series $Q(x,y)$, either for showing its non-D-algebraicity in x and y, or for obtaining an explicit algebraic expression.

2.1. **Showing non D-algebraicity.** Our argument for showing non-D-algebraicity will rely on the fact that we know that the solution $Q(x,y)$ is a generating function of walks in the quadrant, and thus we may control the expansion of $Q(x,0)$ and $Q(0,y)$ around 0.

Lemma 2.4. *Let h be a fraction of $\mathbb{C}(s)$, and assume that the poles of h belong to $\{0,\infty\}$. If $h^{\iota_1} = h$, then there exists a Laurent polynomial $H(x) \in \mathbb{C}[1/x]$ such that $H(x(s)) = h(s)$. Analogously, if $h^{\iota_2} = h$, then there exists a Laurent polynomial $H(y) \in \mathbb{C}[1/y]$ such that $H(y(s)) = h(s)$.*

Proof. Let $h \in \mathbb{C}(s)$ be a function whose poles belong to $\{0,\infty\}$, such that $h^{\iota_1} = h$. The extension $\mathbb{C}(s)/\mathbb{C}(x(s))$ being Galois of Galois group generated by ι_1 by Proposition 1.2, there exists a fraction $H(x) \in \mathbb{C}(x)$ such that $H(x(s)) = h(s)$. Write $H(x) = \frac{U(x)}{V(x)}$ with U and V relatively prime polynomials, V monic, and let $u(s) \overset{\text{def}}{=} U(x(s))$, $v(s) \overset{\text{def}}{=} V(x(s))$. We write the polar divisor $(h)_\infty$ of h in two different ways.

First, we have by the assumption that $(h)_\infty = p \cdot 0 + q \cdot \infty$ for some nonnegative integers p and q. As $\iota_1(0) = \infty$ and $h^{\iota_1} = h$, we conclude that $p = q$, so by Proposition 1.2,

$$(h)_\infty = p \cdot 0 + p \cdot \infty = p \cdot (x(s))_0. \tag{2.5}$$

Moreover, as u and v are polynomials in $x(s)$, we have that $(u)_\infty = \deg_x U(x) \cdot (x(s))_\infty$. and $(v)_\infty = \deg_x V(x) \cdot (x(s))_\infty$. We also have that since the polynomials $U(x)$ and $V(x)$ are relatively prime, Bézout theorem implies the existence of some relation

$$U'(x)U(x) + V'(x)V(x) = 1$$

for $U'(x), V'(x) \in \mathbb{C}[x]$. Thus, by composing the relation with $x(s)$, we see that

$$u'(s)u(s) + v'(s)v(s) = 1.$$

For s_0 a pole of $x(s)$, $U(x(s_0))$ and $V(x(s_0))$ are both nonzero since $U(x)$ and $V(x)$ are polynomials. For s_0 not a pole of $x(s)$, then s_0 is not a pole of u', v', u, v, and we see that it cannot be a zero of both $u(s)$ and $v(s)$. Thus, since

$$\begin{aligned}(h) &= (u)_0 + (v)_\infty - (u)_\infty - (v)_0 \\ &= (u)_0 - (v)_0 + (\deg_x V(x) - \deg_x U(x)) \cdot (x(s))_\infty\end{aligned}$$

and the zeros of u and v don't compensate, we deduce that

$$(h)_\infty = (v)_0 - \min(\deg_x V(x) - \deg_x U(x), 0) \cdot (x(s))_\infty. \tag{2.6}$$

Therefore, equating (2.5) and (2.6), we obtain the conditions

(1) $\deg_x V(x) - \deg_x U(x) \geq 0$ (2) $(v)_0 = p \cdot (x(s))_0$.

Consider $w(s) \overset{\text{def}}{=} \frac{v(s)}{x(s)^p}$. Using (2), we compute its divisor as

$$\begin{aligned}(w) &= (v)_0 - (v)_\infty - p \cdot (x(s))_0 + p \cdot (x(s))_\infty \\ &= (p - \deg_x V(x)) \cdot (x(s))_\infty.\end{aligned}$$

Since $\deg(w) = 0$ (Proposition 1.1), we deduce that $2 \cdot (p - \deg_x V(x)) = 0$, and thus $(w) = 0$, which implies that w is a constant (Proposition 1.1). As $V(x)$ is monic, this implies that $V(x) = x^p$. Moreover condition (1) implies that $\deg_x U(x) \leq \deg_x V(x) = p$. We thus conclude that $H(x) = \frac{U(x)}{V(x)}$ belongs to $\mathbb{C}[1/x]$. The proof for $h^{\iota_2} = h$ is similar. □

Lemma 2.5. *Assume that the following two conditions hold:*

1. *For any pair of solutions (h_1, h_2) of $(E_{\widetilde{\gamma}_1, \widetilde{\gamma}_2, \omega})$, one of the functions h_1 or h_2 has its poles in $\{0, \infty\}$.*
2. *The only solution of $(E'_{\widetilde{\gamma}_1, \widetilde{\gamma}_2})$ is $(0, 0)$.*

Then $Q(x, y)$ is non x-D-algebraic nor y-D-algebraic.

Proof. Assume that $Q(x, y)$ is x-D-algebraic or y-D-algebraic, then by Proposition 1.6 the functions $\widetilde{F}$ and $\widetilde{G}$ are rational. Hence, by Lemma 2.2, the pair $(\widetilde{f}, \widetilde{g})$ satisfies $(E_{\widetilde{\gamma}_1, \widetilde{\gamma}_2, \omega})$ and the pair $(\widetilde{f}_h, \widetilde{g}_h)$ satisfies $(E'_{\widetilde{\gamma}_1, \widetilde{\gamma}_2})$, with $\widetilde{F} = \widetilde{f} + \widetilde{f}_h$ and $\widetilde{G} = \widetilde{g} + \widetilde{g}_h$.

From (1), assume without loss of generality that the poles of $\widetilde{f}$ are in the set $\{0, \infty\}$. Then by Lemma 2.4 applied to $\widetilde{f}$, there exists a Laurent polynomial $f(x) \in \mathbb{C}[1/x]$ such that $f(x(s)) = \widetilde{f}(s)$. Denote by $-d$ the valuation of $f(x)$, so that $P(x) \overset{\text{def}}{=} x^d f(x)$ is a polynomial of degree at most d. From (2), we have that $\widetilde{f}_h = 0$, hence $\widetilde{F}(s) = \widetilde{f}(s)$. Therefore, as $\widetilde{F}(s)$ is a continuation of $\check{F}(s)$, we have for $s \in V$ the equation

$$x(s)Q(x(s), 0) - f(x(s)) = 0.$$

The function $xQ(x, 0) - f(x)$ is meromorphic at $x = 0$, hence by Lemma 1.7 it is zero, so we have the equation

$$x^{d+1}Q(x, 0) = P(x).$$

But we have that $Q(x, 0) = 1 + O(x)$, which is a contradiction because $P(x)$ has degree at most d. □

2.2. **Retrieving D-algebraic solutions.** In the remaining cases, we are able to prove that $Q(x, y)$ is D-algebraic by lifting rational solutions (in $\mathbb{C}(s)$) to the decoupling equations $(E_{\widetilde{\gamma}_1, \widetilde{\gamma}_2, \omega})$ and $(E'_{\widetilde{\gamma}_1, \widetilde{\gamma}_2})$ to algebraic solutions (over $\mathbb{C}(x, y, t)$) of the main functional equation (1.1). Recall that $\omega = \frac{1}{ab}(a + b - ab)$, as defined in (1.2)

Lemma 2.6. *Assume that $\omega = 0$, and that (h_1, h_2) is a nonzero solution to $(E_{\widetilde{\gamma}_1, \widetilde{\gamma}_2, \omega})$ or $(E'_{\widetilde{\gamma}_1, \widetilde{\gamma}_2})$. Then the function h_2 satisfies the identity $\frac{h_2^{\sigma}(s)}{h_2(s)} = \frac{\widetilde{\gamma}^{\iota_1}(s)}{\widetilde{\gamma}(s)}$.*

Proof. Let (h_1, h_2) be a pair solution to either $(E_{\widetilde{\gamma}_1, \widetilde{\gamma}_2, \omega})$ or $(E'_{\widetilde{\gamma}_1, \widetilde{\gamma}_2})$. In both cases, there exists some $\varepsilon \in \{-1, 1\}$ such that $h_1^{\iota_1}(s) = \varepsilon h_1(s)$ and $h_2^{\iota_2}(s) = \varepsilon h_2(s)$.

Analogously to what was done in the first section, we start from the identity

$$\widetilde{\gamma}(s)h_1(s) = -h_2(s). \tag{2.7}$$

Applying ι_1 on both sides of the equation and using the relation $h_1^{\iota_1} = \varepsilon h_1$, we obtain

$$\varepsilon\widetilde{\gamma}^{\iota_1}(s)h_1(s) = -h_2^{\iota_1}(s). \tag{2.8}$$

Eliminating $h_1(s)$ between (2.7) and (2.8), and using the identity $h_2^{\iota_1}(s) = \varepsilon h_2^{\iota_2\iota_1}(s) = \varepsilon h_2^{\sigma}(s)$ shows the claim. ☐

Lemma 2.7. *Assume that $\omega = 0$.*

1. *If $(E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega})$ admits a nonzero solution $(h_1,h_2) \in \mathbb{C}(s)$, then $Q(x,y)$ is rational in x and y (for the fixed t of Proposition 1.5). More precisely, there exist $H_1(z), H_2(z) \in \mathbb{C}(z)$ such that $H_1(x(s)) = h_1(s)$ and $H_2(y(s)) = h_2(s)$, and $\lambda \in \mathbb{C}$ such that*
$$xQ(x,0) = \lambda H_1(x), \qquad yQ(0,y) = \lambda H_2(y).$$
2. *If $(E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2})$ admits a nonzero solution $(h_1,h_2) \in \mathbb{C}(s)$, then $Q(x,y)$ is algebraic over $\mathbb{C}(x,y)$ (for the fixed t of Proposition 1.5). More precisely, there exist $H_1(z), H_2(z) \in \mathbb{C}(z)$ such that $H_1(x(s)) = h_1(s)^2$ and $H_2(y(s)) = h_2(s)^2$, and $\lambda \in \mathbb{C}$ such that*
$$xQ(x,0) = \pm\lambda\sqrt{H_1(x)}, \qquad yQ(0,y) = \pm\lambda\sqrt{H_2(y)}.$$

Proof. Let (h_1,h_2) be a nonzero solution to $(E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega})$ or $(E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2})$. By Lemma 2.6, the function $h_2(s)$ satisfies

$$\frac{h_2^{\sigma}(s)}{h_2(s)} = \frac{\widetilde{\gamma}^{\iota_1}(s)}{\widetilde{\gamma}(s)}. \tag{2.9}$$

Now, considering the function $H(s) \overset{\text{def}}{=} \frac{\widetilde{G}(s)}{h_2(s)}$ (recall that h_2 is nonzero), we see by combining equations (2.9) and (1.13) that $H(qs) = H(s)$. The function $H(s)$ is meromorphic on $\mathbb{C}$ and $|q| \neq 1$, hence $H(s)$ is a constant. Therefore, there exists $\lambda \in \mathbb{C}$ such that $\widetilde{G}(s) = \lambda h_2(s)$. By Equation (1.12), we also deduce that $\widetilde{F}(s) = \lambda h_1(s)$.

We can now prove the two cases of the lemma:

1. If $(h_1(s), h_2(s))$ is a solution to $(E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega})$, then by Proposition 1.3, there exist $H_1(x) \in \mathbb{C}(x)$ and $H_2(y) \in \mathbb{C}(y)$ such that $H_1(x(s)) = h_1(s)$ and $H_2(y(s)) = h_2(s)$. Therefore, $x(s)Q(x(s),0) = \lambda H_1(x(s))$, and $y(s)Q(0,y(s)) = \lambda H_2(y(s))$ for all s in V. Thus, as these functions are meromorphic at $s = 0$, Proposition 1.7 yields $xQ(x,0) = \lambda H_1(x)$ and $yQ(0,y) = \lambda H_2(y)$.
2. If $(h_1(s), h_2(s))$ is a solution to $(E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2})$, then $(h_1(s)^2)^{\iota_1} = h_1(s)^2$ and $(h_2(s)^2)^{\iota_2} = h_2(s)^2$. By Proposition 1.3, there exist $H_1(x) \in \mathbb{C}(x)$ and $H_2(y) \in \mathbb{C}(y)$ such that $H_1(x(s)) = h_1(s)^2$ and $H_2(y(s)) = h_2(s)^2$. Therefore, $x(s)^2Q(x(s),0)^2 = \lambda^2 H_1(x(s))$ and $y(s)^2Q(0,y(s))^2 = \lambda^2 H_2(y(s))$ for all s in V. Thus, as these functions are meromorphic at $s = 0$, Proposition 1.7 yields $x^2Q(x,0)^2 = \lambda^2 H_1(x)$ and $y^2Q(0,y)^2 = \lambda^2 H_2(y)$. ☐

Remark 2.8.

1. The value of λ can be found by evaluating $H_1(x)$ at $x = 0$ (resp. $H_2(y)$ at $y = 0$), and using the fact that $Q(0,0) = 1$.
2. Note that the expressions for $H_1(x)$ and $H_2(y)$ depend *a priori* on the value of the real number t. However, in all the algebraic cases, the fractions $H_1(z)$ and $H_2(z)$ will be fixed fractions of $\mathbb{Q}(d_{i,j}, a, b, t, z)$ analytic at $t = 0$. Thus, we may lift the solutions for $Q(x,0)$ and $Q(0,y)$ as formal power series in t.

■

The classification will go as follows: for every model and parameters, we will show that we are either in the case of application of Lemma 2.5 or Lemma 2.7, by studying the decoupling equations ($E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega}$) and ($E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2}$).

3. DECOUPLING EQUATIONS

In the previous section, we reduced the classification to the study of two decoupling equations ($E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega}$) and ($E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2}$). According to the previously designed strategy, we will investigate both equations separately.

3.1. **Homogeneous equation.** We first handle the homogeneous equation ($E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2}$), which corresponds to the standard case of a *multiplicative decoupling* (like for instance those in [BPFHR25]), in this case of the fraction $\widetilde{\gamma} \overset{\text{def}}{=} \widetilde{\gamma}_1/\widetilde{\gamma}_2$. We want to determine in which cases this equation admits rational solutions. When they exist, we provide them explicitly. Otherwise, we provide two distinct arguments to show the non-existence of rational solutions. The first one is standard, and revolves around a process called *pole propagation*.

Lemma 3.1. *Let $u(s)$ be in $\mathbb{C}(s)$. Assume that there exists a pole (resp. zero) $P \neq \{0,\infty\}$ of $u(s)$ such that for all n in $\mathbb{Z}$ the point $\sigma^n P$ is never a zero (resp. pole) of $u(s)$.*

Then there is no nonzero $h(s) \in \mathbb{C}(s)$ such that $h^\sigma(s) = u(s)h(s)$.

Proof. The result is elementary, see for instance [HS08, Lemma 3.5] for a proof. ☐

We apply this lemma to the homogeneous equation ($E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2}$).

Corollary 3.2. *Assume that there exists $P \neq \{0,\infty\}$ a pole (resp. zero) of $\frac{\widetilde{\gamma}^{\iota_1}}{\widetilde{\gamma}}$ such that for all n in $\mathbb{Z}$ the point $\sigma^n P$ is never a zero (resp. pole) of $\frac{\widetilde{\gamma}^{\iota_1}}{\widetilde{\gamma}}$. Then the equation ($E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2}$) has no nonzero rational solution.*

Proof. Assume that (h_1,h_2) is a nonzero solution to ($E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2}$). Then by Lemma 2.6, the function $h_2(s)$ satisfies the equation $\frac{h_2^\sigma(s)}{h_2(s)} = \frac{\widetilde{\gamma}^{\iota_1}(s)}{\widetilde{\gamma}(s)}$, which by assumption and Lemma 3.1 has no nonzero rational solution, a contradiction. ☐

Unfortunately, this pole propagation technique does not discard all cases where the homogeneous solution has no solution. Mainly, it may happen that there exists a pair of fractions (h_1,h_2) satisfying the linear relation with ${h'_1}^{\iota_1}(s) = \pm h'_1(s)$ and ${h'_2}^{\iota_2}(s) = \pm h'_2(s)$ (we call such relaxed solutions *signed solutions*). In this case, Corollary 3.2 will not apply. However, when such solution with "wrong" signs for either ${h'_1}^{\iota_1}(s)/h'_1(s)$ or ${h'_2}^{\iota_2}(s)/h'_2(s)$ exists, we show that ($E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2}$) has no nonzero rational solution (with the "right" signs).

Lemma 3.3. *Let (h'_1,h'_2) be a nonzero pair satisfying the relation $\widetilde{\gamma}_1(s)h'_1(s) + \widetilde{\gamma}_2(s)h'_2(s) = 0$ with $(h'_1)^{\iota_1} = \pm h'_1$ and $(h'_2)^{\iota_2} = \pm h'_2$. If $(h'_1)^{\iota_1} = h'_1$ or $(h'_2)^{\iota_2} = h'_2$, then Equation ($E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2}$) has no nontrivial rational solution.*

Proof. Assume that (h'_1,h'_2) is such a pair, and let (h_1,h_2) be a pair of rational solutions to ($E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2}$). Then we have the equation

$$\frac{h_1}{h'_1} = \frac{h_2}{h'_2} =: u.$$

Now, by the symmetries of the $h_{1,2}$ and $h'_{1,2}$, we have that $(u^2)^{\iota_1} = (u^2)^{\iota_2} = u^2$. Therefore, the fraction u^2 is fixed by σ, hence $u^2 \in \mathbb{C}$, so $u \in \mathbb{C}$.

Assuming that $(h_1')^{\iota_1} = h_1'$, we obtain that $h_1^{\iota_1} = (uh_1')^{\iota_1} = uh_1' = h_1$. As h_1 satisfies also $h_1^{\iota_1} = -h_1$, we deduce that $h_1 = 0$, and thus that $h_2 = 0$.

Similarly, assuming that $(h_2')^{\iota_2} = h_2'$, we obtain that $(h_1, h_2) = (0,0)$. □

3.2. **Inhomogeneous equation.** To handle the inhomogeneous equation, we will proceed as in Lemma 3.1 by proving a propagation lemma adapted to $(E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega})$ (namely Lemma 3.6 below). Unlike the previous case, this lemma will only restrict the possible poles of h_1 and h_2 to a finite set of points of $\mathbb{P}^1$, for (h_1,h_2) a solution. In most cases, we will be able to show that one of $h_{1,2}$ must have its poles restricted to the set $\{0,\infty\}$, which is one of the requirements of Lemma 2.5 which shows non-D-algebraicity of $Q(x,y)$ in x and y.

Before we state and prove the propagation lemma for the inhomogeneous equation, we recall two easy facts on poles of rational maps on a curve. The first fact concerns the relations between the poles of two functions related by a linear equation.

Lemma 3.4. *Assume that $(h_1,h_2) \in \mathbb{C}(s)^2$ satisfies the relation $u_1h_1 + u_2h_2 + u_3 = 0$ for some u_1, u_2, u_3 in $\mathbb{C}(s)$. If P is a pole of h_1 not in $\{(u_1)_0, (u_2)_\infty, (u_3)_\infty\}$, then it is a pole of h_2.*

Proof. Assume that P is a pole of $h_1(s)$. Since it is not a zero of $u_1(s)$, it is a pole of $u_1(s)h_1(s)$. By the relation, P is a pole of $u_2(s)h_2(s) + u_3(s)$. Since P is not a pole of $u_3(s)$, it is a pole of $u_2(s)h_2(s)$, and because it is not a pole of $u_2(s)$, this implies that P is a pole of $h_2(s)$. □

The second standard fact concerns the poles of a function stable by automorphisms.

Lemma 3.5. *Let h be in $\mathbb{C}(s)$, and τ an automorphism of $\mathbb{P}^1$. If P is a pole (resp. zero) of h, then $\tau^{-1}P$ is a pole (resp. zero) of h^τ. In particular, if $h^\tau = \lambda h$ for some nonzero $\lambda \in \mathbb{C}$, then the set of poles (resp. zeros) of h is stable under the action of τ.*

Proof. Assume that P given by $s_0 \in \mathbb{P}^1$ is a zero of h. Then $h^\tau(\tau^{-1}(s_0)) = h(\tau(\tau^{-1}(s_0))) = h(s_0) = 0$. □

We now state the announced propagation lemma, specific to $(E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega})$. It depends on the divisors of the coefficients of this equation. Recall from Proposition 1.4 that

$$(\star) \qquad (\widetilde{\gamma}_1) = P_1 + P_2 - 0 - \infty, \qquad (\widetilde{\gamma}_2) = P_3 + P_4 - 0 - \infty;$$

and that $\omega = 1 - A - B$ is a constant. We define four finite sets $\mathcal{L}_1^-, \mathcal{L}_1^+, \mathcal{L}_2^-, \mathcal{L}_2^+ \subset \mathbb{P}^1$ as follows:

$$(3.1) \qquad \begin{aligned} \mathcal{L}_1^- &\overset{\text{def}}{=} \{P_1, P_2, \iota_2 P_3, \iota_2 P_4\}, & \mathcal{L}_1^+ &\overset{\text{def}}{=} \iota_1 \mathcal{L}_1^- = \{\iota_1 P_1, \iota_1 P_2, \sigma^{-1} P_3, \sigma^{-1} P_4\}, \\ \mathcal{L}_2^- &\overset{\text{def}}{=} \{\sigma P_1, \sigma P_2, \iota_2 P_3, \iota_2 P_4\}, & \mathcal{L}_2^+ &\overset{\text{def}}{=} \iota_2 \mathcal{L}_2^+ = \{\iota_1 P_1, \iota_1 P_2, P_3, P_4\}. \end{aligned}$$

We call the elements of the sets $\mathcal{L}_{1,2}^{+,-}$ the *critical points* of equation $(E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega})$.

Lemma 3.6 (Pole propagation)**.** *Let $(h_1,h_2) \in \mathbb{C}(s)^2$ be a solution to $(E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega})$.*

Let P be a pole of h_1 distinct from 0, ∞.

(i) If $P \notin \mathcal{L}_1^-$, then $\sigma^{-1}P$ is a pole of h_1.
(ii) If $P \notin \mathcal{L}_1^+$, then σP is a pole of h_1.

Let P be a pole of h_2 distinct from 0, ∞.

(i') If $P \notin \mathcal{L}_2^-$, then $\sigma^{-1}P$ is a pole of h_2.
(ii') If $P \notin \mathcal{L}_2^+$, then σP is a pole of h_2.

Proof. We prove (i). Let P be a pole of h_1 distinct from 0, ∞ and not in $\mathcal{L}_1^-$. Because $P \notin \{P_1, P_2, 0, \infty\} = \{(\widetilde{\gamma}_1)_0, (\widetilde{\gamma}_2)_\infty, (\omega)_\infty\}$ (see $(\star)$ above), this implies by Lemma 3.4 that P is also a pole of h_2. Now, $h_2^{\iota_2} = h_2$, hence $\iota_2 P$ is a pole of h_2 by Lemma 3.5. Now, $P \notin \{\iota_2 P_3, \iota_2 P_4, 0, \infty\}$ hence $\iota_2 P \notin \{P_3, P_4, 0, \infty\} = \{(\widetilde{\gamma}_2)_0, (\widetilde{\gamma}_1)_\infty, (\omega)_\infty\}$, so by Lemma 3.4 the point $\iota_2 P$ is a pole of h_1. Finally, $h_1^{\iota_1} = h_1$, hence $\sigma^{-1} P = \iota_1(\iota_2 P)$ is a pole of h_1.

We now prove (ii). Let P be a pole of h_1 distinct from 0, ∞ and not in $\mathcal{L}_1^+$. Since $P \notin \mathcal{L}_1^+ = \iota_1 \mathcal{L}_1^-$, then $\iota_1 P \notin \mathcal{L}_1^-$, and $\iota_1 P \neq 0, \infty$. By (i), this implies that $\sigma^{-1}(\iota_1 P) = \iota_1(\sigma P)$ is a pole of h_1. As $h_1^{\iota_1} = h_1$, this implies that σP is a pole of h_1 by Lemma 3.5.

The proofs of (i') and (ii') are similar. ☐

Using the critical points, we may thus describe the possible poles of rational solutions (h_1, h_2) to $(E_{\widetilde{\gamma}_1, \widetilde{\gamma}_2, \omega})$.

Lemma 3.7. *Let $(h_1, h_2) \in \mathbb{C}(s)^2$ be a solution to $(E_{\widetilde{\gamma}_1, \widetilde{\gamma}_2, \omega})$. If P is a pole of h_1 distinct from 0 or ∞, then there exist two integers $m, n \geq 0$ such that $\sigma^{-m} P \in \mathcal{L}_1^-$ and $\sigma^n P \in \mathcal{L}_1^+$. Likewise, if P is a pole of h_2 distinct from 0 or ∞, then there exist two integers $m, n \geq 0$ such that $\sigma^{-m} P \in \mathcal{L}_2^-$ and $\sigma^n P \in \mathcal{L}_2^+$.*

Proof. Assume for the sake of contradiction that $P \neq 0, \infty$ is a pole of h_1 satisfying $\sigma^n P \notin \mathcal{L}_1^+$ for all $n \geq 0$. Then by induction and Lemma 3.6, we show that $\sigma^n P$ is a pole of h_1 for all $n \geq 0$. Since P is distinct from 0 or ∞, the orbit $(\sigma^n P)_{n \geq 0}$ is infinite, hence the fraction h_1 has an infinite number of poles, a contradiction. The other points are proved in a similar fashion. ☐

3.3. ***σ*-distance.** Thanks to pole propagation, Lemma 3.7 allows us to locate the possible poles of h_1 and h_2 for a rational solution (h_1, h_2) to $(E_{\widetilde{\gamma}_1, \widetilde{\gamma}_2, \omega})$. Similarly, Lemma 3.1 gives a sufficient condition for proving that $(E'_{\widetilde{\gamma}_1, \widetilde{\gamma}_2})$ has no nonzero rational solution (h_1', h_2'). These two lemmas thus give conditions to the existence of solutions to decoupling equations based on the relations between the points of the finite sets $\mathcal{L}_{1,2}^{\pm}$. These relations are captured by a signed distance called the σ-*distance*, that we introduce to compare two points of $\mathbb{P}^1$ with respect to the action of the group $< \sigma >$. This will give us numerical data from which we will conduct the classification.

Definition/Proposition 3.8. Let P and P' be two points of $\mathbb{P}^1$ distinct from 0 and ∞. We define the σ-*distance* $\delta(P, P')$ of the points P and P' as follows:

- If there exists an integer $n \in \mathbb{Z}$ such that $\sigma^n P = P'$, then n is unique, and we define $\delta(P, P') = n$.
- Otherwise, if no such integer exists, we define $\delta(P, P') = \perp$.

Proof. We just need to show the uniqueness. Assume that $\sigma^n P = \sigma^m P$ for two integers m and n. Then $\sigma^{n-m} P = P$, which is possible if and only if if $n = m$ because 0 and ∞ are the only periodic points of the action of σ on $\mathbb{P}^1$ (indeed, if $q^n s = s$ with $n \geq 1$, then $s = 0$ or $s = \infty$, for $q^n \neq 1$). ☐

The σ-distance satisfies the standard arithmetic properties that one would expect from a signed distance.

Proposition 3.9. *With the convention that $n + \perp = \perp$ for every integer n, and that $\perp = -\perp$, the σ-distance satisfies the following properties for points P, P' and P'' distinct from 0 or ∞:*

(i) $\delta(P, P') = -\delta(P', P)$,
(ii) $\delta(P, P') + \delta(P', P'') = \delta(P, P'')$ *if* $\delta(P, P')$ *and* $\delta(P', P'')$ *are finite,*
(iii) $\delta(P, \sigma(P')) = \delta(P, P') + 1$,
(iv) $\delta(P, P') = \delta(\iota_1 P', \iota_1 P) = \delta(\iota_2 P', \iota_2 P)$.

Proof. The proofs of (i), (ii) and (iii) are straightforward, hence we focus on the proof of (iv). Assume first that $\sigma^n P = P'$ for some integer n. Recall that $\sigma = \iota_2\iota_1$ with $\iota_1^2 = \iota_2^2 = \mathrm{id}$, so it is easy to see that $\iota_1\sigma^n = \sigma^{-n}\iota_1$. Thus, $\sigma^{-n}(\iota_1 P) = \iota_1(\sigma^n P) = \iota_1 P'$, which implies that $\delta(\iota_1 P, \iota_1 P')$ is finite, being equal to $-n = -\delta(P,P') = \delta(P',P)$. The application ι_1 is an involution, thus if $\delta(P,P') = \perp$, then we also have $\delta(\iota_1 P, \iota_1 P') = \perp$. ☐

We are going to determine the values of $\delta(P,P')$ for all $(P,P') \in \mathcal{L}_1^- \times \mathcal{L}_1^+$ and $(P,P') \in \mathcal{L}_2^- \times \mathcal{L}_2^+$. These values are compiled respectively in the matrices M_1 and M_2 in $\mathcal{M}_4(\mathbb{Z} \cup \{\perp\})$, the lines ($\mathcal{L}_{1,2}^-$) and columns ($\mathcal{L}_{1,2}^+$) being ordered as in (3.1). More precisely, their entries are organized as follows:

$$M_1 \overset{\text{def}}{=} \begin{pmatrix} \delta(P_1,\iota_1P_1) & \delta(P_1,\iota_1P_2) & \delta(P_1,\sigma^{-1}P_3) & \delta(P_1,\sigma^{-1}P_4) \\ \delta(P_2,\iota_1P_1) & \delta(P_2,\iota_1P_2) & \delta(P_2,\sigma^{-1}P_3) & \delta(P_2,\sigma^{-1}P_4) \\ \delta(\iota_2P_3,\iota_1P_1) & \delta(\iota_2P_3,\iota_1P_2) & \delta(\iota_2P_3,\sigma^{-1}P_3) & \delta(\iota_2P_3,\sigma^{-1}P_4) \\ \delta(\iota_2P_4,\iota_1P_1) & \delta(\iota_2P_4,\iota_1P_2) & \delta(\iota_2P_4,\sigma^{-1}P_3) & \delta(\iota_2P_4,\sigma^{-1}P_4) \end{pmatrix},$$

$$(3.2)\qquad M_2 \overset{\text{def}}{=} \begin{pmatrix} \delta(\sigma P_1,\iota_1P_1) & \delta(\sigma P_1,\iota_1P_2) & \delta(\sigma P_1,P_3) & \delta(\sigma P_1,P_4) \\ \delta(\sigma P_2,\iota_1P_1) & \delta(\sigma P_2,\iota_1P_2) & \delta(\sigma P_2,P_3) & \delta(\sigma P_2,P_4) \\ \delta(\iota_2P_3,\iota_1P_1) & \delta(\iota_2P_3,\iota_1P_2) & \delta(\iota_2P_3,P_3) & \delta(\iota_2P_3,P_4) \\ \delta(\iota_2P_4,\iota_1P_1) & \delta(\iota_2P_4,\iota_1P_2) & \delta(\iota_2P_4,P_3) & \delta(\iota_2P_4,P_4) \end{pmatrix}.$$

Proposition 3.10. *The matrices M_1 and M_2 satisfy the following relations:*

(i) $M_1^T = M_1$ *and* $M_2^T = M_2$,

(ii) $M_2 = M_1 + \begin{pmatrix} -J_2 & 0 \\ 0 & J_2 \end{pmatrix}$ *where* $J_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

Proof. These are straightforward applications of Proposition 3.9. ☐

Thanks to the above proposition, it is only required to compute the σ-distances of $10 = 4+3+2+1$ pairs of points, namely those above the diagonal of M_1. Note that the sets $\mathcal{L}_{1,2}^{\pm}$, and thus the matrix M_1 depend on the set of steps $\mathcal{S}_i$ of the model and weights $d_{i,j}$, A and B. We finish this subsection by giving two lemmas to exploit these matrices.

Lemma 3.11. *Let (h_1,h_2) be a pair of rational solutions to ($E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega}$).*

(i) If all the entries of M_1 are in $\mathbb{Z}^- \cup \{\perp\}$, then the poles of $h_1(s)$ belong to $\{0,\infty\}$.
(ii) If all the entries of M_2 are in $\mathbb{Z}^- \cup \{\perp\}$, then the poles of $h_2(s)$ belong to $\{0,\infty\}$.

Proof. Let (h_1,h_2) be a pair of rational solutions to ($E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega}$). We prove (i). Assume that all the entries of M_1 are in $\mathbb{Z}^- \cup \{\perp\}$. If $P \neq \{0,\infty\}$ is a pole of h_1, then by Lemma 3.7, there exist $m,n \geq 0$ such that $\sigma^{-m}P =: Q^- \in \mathcal{L}_1^-$ and $\sigma^n P =: Q^+ \in \mathcal{L}_1^+$. But then by (ii) of Proposition 3.9, one has $\delta(Q^-,Q^+) = \delta(Q^-,P) + \delta(P,Q^+) = n+m \geq 0$, a contradiction since $\delta(Q^-,Q^+)$ is an entry of M_1. Thus, h_1 has no poles besides 0 and ∞. The proof of point (ii) is similar. ☐

Lemma 3.12. *If one of the rows of M_1 consists of $\perp$'s only, then ($E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2}$) has no nonzero rational solution.*

Proof. From Proposition 1.4, we may write the divisor of $\widetilde{\gamma}^{\iota_1}/\widetilde{\gamma}$ as

$$(\widetilde{\gamma}^{\iota_1}/\widetilde{\gamma}) = \iota_1 P_1 + \iota_1 P_2 + P_3 + P_4 - P_1 - P_2 - \iota_1 P_3 - \iota_1 P_4.$$

Assume that there exists $Q^- \in \mathcal{L}_1^-$ such that for all $Q^+ \in \mathcal{L}_1^+$ one has $\delta(Q^-, Q^+) = \perp$ (Q^- labels the row M_1 consisting of $\perp$'s only). Then there exists an integer k such that $Q' \overset{\text{def}}{=} \sigma^k Q^-$ is a pole of $\widetilde{\gamma}^{\iota_1}/\widetilde{\gamma}$, namely $k = 0$ for $Q^- \in \{P_1, P_2\}$ or $k = -1$ for $Q^- \in \{\iota_2 P_3, \iota_2 P_4\}$.

- The point Q' is a pole of $\widetilde{\gamma}^{\iota_1}/\widetilde{\gamma}$.
- The point $\sigma^n Q'$ is never a zero of $\widetilde{\gamma}^{\iota_1}/\widetilde{\gamma}$. Indeed, if it were the case, then $Q^+ \overset{\text{def}}{=} \sigma^m Q'$ would belong to $\mathcal{L}_1^+$, either for $m = n$ if $\sigma^n Q' \in \{\iota_1 P_1, \iota_1 P_2\}$, or $m = n-1$ if $\sigma^n Q' \in \{P_3, P_4\}$. The point (iii) of Proposition 3.9 would then imply that

  $$\delta(Q^-, Q^+) = \delta(Q^-, Q') + \delta(Q', Q^+) = k + m,$$

  while the row of M_1 corresponding to Q^- consisting of $\perp$'s only implies that $\delta(Q^-, Q^+) = \perp$, a contradiction.

Therefore by Corollary 3.2, $(E'_{\widetilde{\gamma}_1, \widetilde{\gamma}_2})$ has no nonzero rational solution. ☐

4. COMPUTING THE σ-DISTANCE

Denote by $\mathbb{F}$ the field $\mathbb{Q}(d_{i,j}, a, b)$. In this section, we describe a heuristic to decide given two points P and P' of $\mathbb{P}^1$ if there exists an integer n such that $\sigma^n(P) = P'$. In other words, for two points P and P' of $\mathbb{P}^1$, the goal is to compute the σ-distance $\delta(P, P')$ of Definition 3.8. When we restrict to the points that originate from zeros of the fractions $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$, the σ-distance is computable. The main argument used here, and already exploited for instance in [BM10], is based on *valuations*.

Definition/Proposition 4.1. There exists an embedding (a $\mathbb{F}$-algebra homomorphism) $\psi : \overline{\mathbb{F}(t)} \longrightarrow \mathbb{C}^{frac}((T))$ where $\mathbb{C}^{frac}((T))$ is the field of formal Puiseux series over $\mathbb{C}$ in the formal variable T. [1] We fix once and for all this embedding ψ. As a result, for any $u \in \overline{\mathbb{F}(t)}$, we define its *valuation* $v(u) \in \mathbb{Q}$ to be the valuation in the variable T of $\psi(u)$ (the valuation of 0 being defined as $+\infty$).

Proof. Consider the embedding $\psi_5 : \mathbb{F}(t) \longrightarrow \mathbb{C}^{frac}((T))$ defined as the composition

$$\mathbb{F}(t) \xrightarrow{\psi_1} \mathbb{F}(T) \xrightarrow{\psi_2} \mathbb{C}(T) \xrightarrow{\psi_3} \mathbb{C}((T)) \xrightarrow{\psi_4} \mathbb{C}^{frac}((T))$$

where the embedding $\psi_1 : t \mapsto T$ is the isomorphism between $\mathbb{F}(t)$ and $\mathbb{F}(T)$ as t is transcendental over $\mathbb{F}$; the embedding ψ_2 is the map induced by the inclusion $\mathbb{F} \subset \mathbb{C}$; the embedding ψ_3 is the map from $\mathbb{C}(T)$ into the field of Laurent series $\mathbb{C}((T))$; and the embedding ψ_4 is an arbitrary embedding of $\mathbb{C}((T))$ into $\mathbb{C}^{frac}((T))$ its algebraic closure (this follows from the Newton-Puiseux theorem since $\mathbb{C}$ is algebraically closed of characteristic zero [Sta23, Th. 6.1.5]). As the field $\mathbb{C}^{frac}((T))$ is algebraically closed, the embedding ψ_5 admits an extension to an embedding $\psi : \overline{\mathbb{F}(t)} \longrightarrow \mathbb{C}^{frac}((T))$ ([Lan02, Th. V.2.8]). ☐

Definition 4.2. Let $P \in \mathbb{P}^1 \smallsetminus \{0, \infty\}$ such that $\phi(P) = ([1 : x_1], [1 : y_1]) \in \mathbb{P}^1 \times \mathbb{P}^1$ with $x_1, y_1 \in \overline{\mathbb{F}(t)}$. Then define the *bivaluation* of P to be $v(P) \overset{\text{def}}{=} (v(x_1), v(y_1))$.

---

[1] Although the real number t is transcendental over $\mathbb{F}$, we do not directly consider the field of formal Puiseux series in t to avoid conflicts of notation with the usual sum of complex numbers.

Lemma 4.3. *Let* $H(x,y) \in \mathbb{F}(t)(x,y)$ *be a fraction such that* $h(s) \overset{\text{def}}{=} H(x(s),y(s)) \in \mathbb{C}(s)$ *is well defined. If* $P \in \mathbb{P}^1 \setminus \{0,\infty\}$ *is a pole or zero of* h*, then the point* $\phi(P) = ([1:x_1],[1:y_1]) \in \overline{E_t} \subset \mathbb{P}^1 \times \mathbb{P}^1$ *is distinct from* $\Omega = (0,0)$*, and* x_1 *and* y_1 *are algebraic over* $\mathbb{F}(t)$*.*

Proof. Let s_0 be the coordinate in $\mathbb{P}^1$ of P. By assumption, $s_0 \neq 0,\infty$, hence Proposition 1.2 yields $x(s_0) \neq 0$ and $y(s_0) \neq 0$. Moreover, the functions $x(s)$ and $y(s)$ belong to $\overline{\mathbb{F}(t)}(s)$. Hence, $h(s) \in \overline{\mathbb{F}(t)}(s)$, thus if s_0 is a pole or zero of $h(s)$, then s_0 belongs to $\overline{\mathbb{F}(t)}$. Therefore, so do $x_1 = \frac{1}{x(s_0)}$ and $y_1 = \frac{1}{y(s_0)}$. ☐

The above lemma allows us to talk about the bivaluations of the zeros of $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ (the points P_i defined in Proposition 1.4). Now, recall the expressions for ι_1 and ι_2 on $\overline{E_t} \subset \mathbb{P}^1 \times \mathbb{P}^1$ of (1.4) and (1.5):

$$\begin{aligned}\iota_1([1:x_1],[1:y_1]) &= \left([1:x_1],\left[1:\frac{d_{-1,1}x_1^2+d_{0,1}x_1+d_{1,1}}{d_{1,-1}y_1}\right]\right),\\ \iota_2([1:x_1],[1:y_1]) &= \left(\left[1:\frac{d_{1,-1}y_1^2+d_{1,0}y_1+d_{1,1}}{d_{-1,1}x_1}\right],[1:y_1]\right).\end{aligned} \tag{4.1}$$

Hence, for a point $P \in \mathbb{P}^1 \setminus \{0,\infty\}$, the homogeneous coordinates at infinity of $\phi(\iota_1 P) = \iota_1(\phi(P))$ and $\phi(\iota_2 P) = \iota_2(\phi(P))$ (Proposition 1.2) are explicit rational functions in the coordinates of $\phi(P)$. Hence, if the coordinates x_1 and y_1 of $\phi(P)$ are in $\overline{\mathbb{F}(t)}$, then so are the coordinates of $\phi(\iota_1 P)$ and $\phi(\iota_2 P)$. Thus, all the points of the sets $\mathcal{L}^{\pm}_{1,2}$ defined in (3.1) admit a bivaluation. This raises the possibility of keeping track of the successive bivaluations of the points $\sigma^n(P)$ for all integers n and $P \in \mathcal{L}^{\pm}_{1,2}$. It turns out that in most cases, the bivaluations of $\sigma(P)$ and $\sigma^{-1}(P)$ depend only on the bivaluation of P.

Lemma 4.4. *Let* P *be in* $\mathbb{P}^1$ *with* $\phi(P) = ([1:x_1],[1:y_1])$ *and* $x_1, y_1 \in \overline{\mathbb{F}(t)}$*, and let* (i,j) *be* $v(P)$ *the bivaluation of* P*.*

1. *If* $i < 0$ *then* $v(\iota_1(P)) = (i, 2i-j)$*.*
2. *If* $j < 0$ *then* $v(\iota_2(P)) = (2j-i, j)$*.*

Proof. We show (1). Write $\iota_1([1:x_1],[1:y_1]) = ([1:x_1],[1:y_1'])$ (as read in (4.1)). If $v(x_1) < 0$, then since $d_{-1,1} \neq 0$, we have $v(d_{-1,1}x_1^2) = 2v(x_1) = 2i$ and $v(d_{0,1}x_1 + d_{1,1}) \geq v(x_1) > 2v(x_1) = 2i$. Thus, the numerator of y_1' has valuation $2i$. Moreover, since $d_{1,-1} \neq 0$, we compute the valuation of the denominator of y_1' as $v(d_{1,-1}y_1) = v(y_1) = j$, hence the result. The proof of (2) is similar. ☐

Lemma 4.5. *Let* $P \in \mathbb{P}^1$ *with* $\phi(P) = ([1:x_1],[1:y_1])$ *and* $x_1, y_1 \in \overline{\mathbb{F}(t)}$*, and let* $v(P) = (i,j)$ *and* $\delta = |i-j|$*.*

1. *If* $i < j < 0$ *then* $v(\sigma^k(P)) = (i-2\delta k, j-2\delta k)$ *for all* $k \geq 0$*,*
2. *If* $j < i < 0$ *then* $v(\sigma^{-k}(P)) = (i-2\delta k, j-2\delta k)$ *for all* $k \geq 0$*.*

Proof. We prove (1). Assume that $\phi(P) = ([1:x_1],[1:y_1])$ with $x_1, y_1 \in \overline{\mathbb{F}(t)}$ and $(i,j) \overset{\text{def}}{=} v(P) < 0$. We first compute the bivaluation of $\iota_1 P$. As $i < 0$, Lemma 4.4 asserts that $v(\iota_1(P))$ is completely determined by i and j. Thus,

$$v(\iota_1 P) = (i, 2i-j) = (i, j+2(i-j)) = (i, j-2\delta) \text{ since } i < j.$$

As $j - 2\delta < 0$, the bivaluation of $\sigma(P) = \iota_2(\iota_1 P)$ is also completely determined by i and j by Lemma 4.4, and thus

$$v(\sigma P) = (2(j-2\delta) - i, j - 2\delta) = (3i - 2j, j - 2\delta) = (i - 2\delta, j - 2\delta).$$

We thus proved that if $v(P) = (i,j)$ with $i < j < 0$, then $v(\sigma P) = (i - 2\delta, j - 2\delta)$. An easy induction completes the proof. The proof of the second point is similar. $\square$

As the classification of the nature of $Q(x,y)$ given a weighted model depends on the matrices M_1 and M_2 defined in (3.2), we need to compute $\delta(Q^-, Q^+)$ for two points $Q^- \in \mathcal{L}_1^-$ and $Q^+ \in \mathcal{L}_1^+$ for these weights $d_{i,j}$, a, b. Thus, we are able to determine the tables of Appendix A. Note that from Lemma 4.3, both points Q^- and Q^+, with elements of their orbit under σ have a bivaluation. We follow the following method.

(1) Compute $\sigma^n Q^-$ for $n \in \{-2,-1,0,1,2\}$. It happens in all cases (when $Q^- \in \mathcal{L}_1^-$) that $v(\sigma^{-2}(Q^-)) = (i,j)$ with $j < i < 0$ and $v(\sigma^2(Q^-)) = (i,j)$ with $i < j < 0$. Thus, Lemma 4.5 allows us to determine the sequence of bivaluations of $\sigma^n(Q^-)$ for all $n \in \mathbb{Z}$.

(2) Determine the bivaluation (i', j') of the point Q^+.
  - If one point of the orbit of Q^- has bivaluation (i', j') (which we may decide, see the above point), then there are a finite number of n such that $v(\sigma^n(Q^-)) = (i', j')$. For each of these n, check if $\sigma^n(Q^-) = Q^+$. If one of these n works, then $\delta(Q^-, Q^+) = n$.
  - Otherwise, if the bivaluation (i', j') does not appear in the orbit of Q^- or no n works, then $\delta(Q^-, Q^+) = \perp$.

Of course, the space of parameters a, b, $d_{i,j}$ is infinite. Hence, the actual procedure adds the following level of complexity: the bivaluation of a point Q depends on an algebraic condition on the parameters. Thus, we need to explore all the possible bivaluations according to these parameters (for the points Q^- and Q^+). The core of the procedure stays the same. Instead of giving a dry algorithm, we expand below an example.

Example 4.6. We consider the set of steps $\mathcal{S}_1$ of Figure 1.2. In this example, we show how to construct the entry $\delta(P_2, \iota_1 P_2)$ of Table A.1, depending on the weights $d_{0,1}$, $d_{1,-1}$, $d_{-1,1}$, A and B.

(1) The first step consists in computing the bivaluation of P_2 depending on the weights $d_{i,j}$, A and B. We first compute the coordinates of $\phi(P_2)$ for generic $d_{i,j}$, A and B:

$$\phi(P_2) = \left(\left[1 : \frac{-d_{0,1} d_{1,-1} t^2}{d_{1,-1} d_{-1,1} t^2 + A^2 - A} \right], \left[1 : \frac{-A d_{0,1} t}{d_{1,-1} d_{-1,1} t^2 + A^2 - A} \right] \right).$$

The Laurent series expansions of the coordinates of P_2 at $t = 0$ are as follows

$$\phi(P_2) = \left(\left[1 : \frac{-d_{0,1} d_{1,-1} t^2}{A(A-1)} + O(t^3) \right], \left[1 : \frac{-A d_{0,1} t}{A(A-1)} + O(t^2) \right] \right).$$

We thus notice that when $A \neq 0$, the bivaluation of P_2 is $v(P_2) = (2,1)$ (the weights $d_{i,j}$ are nonzero, and note that as $A = 1 - \frac{1}{a}$, then A cannot be equal to 1). Otherwise, in the case $A = 0$, we find $v(P_2) = (0, \infty)$. We now compute their orbits in these two cases.

(a) Assume that $A \neq 0$, so that $v(P_2) = (2,1)$. We check by computing the points $(\sigma^i P_2)_{-2 \leq i \leq 2}$ that their bivaluations do not depend on A as long at it is nonzero, nor the weights $d_{1,-1}$, $d_{-1,1}$ and $d_{0,1}$, and that they are equal to

$$\cdots \to_\sigma (-2,-3) \to_\sigma (0,-1) \to_\sigma v(P_2) = (2,1) \to_\sigma (0,1) \to_\sigma (-2,-1) \to_\sigma \cdots. \tag{4.2}$$

The remaining parts of the above sequence may be continued using Lemma 4.5. Indeed, $v(\sigma^{-2}P_2) = (-2,-3)$, hence $\sigma^{-2}P_2$ satisfies condition (2) of Lemma 4.5, hence we know that whatever the value of $A \neq 0$, one has $v(\sigma^{-k-2}P_2) = (-2-2k, -3-2k)$. Similarly, $v(\sigma^2 P_2) = (-2,-1)$ satisfies condition (1) of Lemma 4.5, hence we deduce that $v(\sigma^{k+2}P_2) = (-2+2k, -1+2k)$ regardless of the values of the weights $d_{i,j}$ and $A \neq 0$.

(b) For $A = 0$, using the same technique, we compute the sequence of bivaluations for $(\sigma^i P_2)_{-3\leq i\leq 1}$ and $A = 0$ (with $v(P_2) = (0,\infty)$):

$$\cdots \to_\sigma (-2,-3) \to_\sigma (0,-1) \to_\sigma (\infty,\infty) \to_\sigma v(P_2) = (0,\infty) \to_\sigma (-2,-1) \to_\sigma \dots, \tag{4.3}$$

Again, the remaining of the above sequence may be continued using Lemma 4.5.

(2) We now compute $\delta(P_2, \iota_1 P_2)$, for $A = 0$ and $A \neq 0$. We first compute the coordinates of $\phi(\iota_1 P_2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ for generic values of the weights $d_{i,j}$, A and B:

$$\phi(\iota_1 P_2) = \left(\left[1 : -\frac{d_{0,1}d_{1,-1}t^2}{d_{1,-1}d_{-1,1}t^2 + A^2 - A} \right], \left[1 : \frac{d_{0,1}t(A-1)}{d_{1,-1}d_{-1,1}t^2 + A^2 - A} \right] \right).$$

We find that

$$v(\iota_1 P_2) = \begin{cases} (2,1) & \text{if } A \neq 0 \\ (0,-1) & \text{if } A = 0 \end{cases}.$$

(a) For $A \neq 0$, since $v(\iota_1 P_2) = (2,1)$, we see by looking at (4.2) that if $\iota_1 P_2$ belongs to the orbit of P_2, then $\iota_1 P_2 = P_2$. This condition is satisfied if and only if $A = \frac{1}{2}$, and then $\delta(P_2, \iota_1 P_2) = 0$. Otherwise, $\delta(P_2, \iota_1 P_2) = \perp$.

(b) For $A = 0$, since $v(\iota_1 P_2) = (0,-1)$, we see by looking at (4.3) that if $\iota_1 P_2$ belongs to the orbit of P_2, then $\iota_1 P_2 = \sigma^{-2}P_2$. This is always the case for any weighting $d_{i,j}$, thus $\delta(P_2, \iota_1 P_2) = -2$.

We thus compute the corresponding entry of Table A.1

$$\delta(P_2, \iota_1 P_2) = \begin{cases} 0 & \text{if } A = \frac{1}{2} \\ -2 & \text{if } A = 0 \\ \perp & \text{otherwise.} \end{cases}$$

The other entries are computed in the same way. ∎

Using this procedure, we manage to compute the entries of M_1 as defined in (3.2) for each set of steps $\mathcal{S}_i$. These matrices are put in Appendix A.

5. Classification

We are now geared to prove the classification.

5.1. Some decouplings and homogeneous solutions.

We first determine for which supports and weights the functional equation ($E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2}$) admits a nonzero rational solution. The following two computational lemmas will be used to exhibit particular signed solutions to ($E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2}$). These statements are checked in the Maple worksheet for this section. Recall the definitions of $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ in (1.6).

Lemma 5.1. *Assume that $d_{1,1} = 0$ (supports $\mathcal{S}_1$ and $\mathcal{S}_2$). Write $x = x(s)$ and $y = y(s)$, and let $\lambda \in \mathbb{C}$. Then the following identities hold:*

(i) $u_\lambda \overset{\text{def}}{=} (1-\lambda) - td_{1,0}x - td_{1,-1}\frac{x}{y} = -(\lambda - td_{0,1}y - td_{-1,1}\frac{y}{x})$,

(ii) $(\lambda - A + x\widetilde{\gamma}_1)\,u_\lambda = \lambda(1-\lambda) - t^2 d_{1,-1}d_{-1,1} - (\lambda t d_{1,0} + t^2 d_{1,-1}d_{0,1})x$,

(iii) $-(1-\lambda - B + y\widetilde{\gamma}_2)\,u_\lambda = \lambda(1-\lambda) - t^2 d_{1,-1}d_{-1,1} - ((1-\lambda)td_{0,1} + t^2 d_{-1,1}d_{1,0})y$.

Lemma 5.2. *Assume that* $d_{1,0} = d_{0,1} = 0$ *(support* $\mathcal{S}_3$*). Write* $x = x(s)$ *and* $y = y(s)$*. Then the following identities hold:*

(i) $(\frac{1}{2} - A + x\widetilde{\gamma}_1)^2 = \frac{1}{4} - d_{1,-1}d_{-1,1}t^2 - d_{1,1}d_{1,-1}t^2x^2$,

(ii) $(\frac{1}{2} - B + y\widetilde{\gamma}_2)^2 = \frac{1}{4} - d_{1,-1}d_{-1,1}t^2 - d_{1,1}d_{-1,1}t^2y^2$.

Using these computations along with Lemma 3.3, and the tables of Section 4 along with Lemma 3.12, we build below Table 5.1[2] which describes the cases where the homogeneous equation $(E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2})$ has solutions. Its entries are read as follows:

- Either the entry has the form $(\perp, i)$, meaning that row i of M_1 is made of $\perp$'s. In this case, the application of Lemma 3.12 shows that there is no nonzero solution to $(E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2})$.
- Either the entry has the form $(\varepsilon_1, \varepsilon_2) \in (\pm, \pm)$, which means that there is a nonzero pair of fractions (h_1', h_2') satisfying the relation $\widetilde{\gamma}_1(s)h_1'(s) + \widetilde{\gamma}_2(s)h_2'(s) = 0$ with $(h_1')^{\iota_1}/h_1' = \varepsilon_1$ and $(h_2')^{\iota_2}/h_2' = \varepsilon_2$ (a *signed solution* to $(E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2})$). If $(\varepsilon_1, \varepsilon_2) \neq (-,-)$, then the application of Lemma 3.3 shows that there is no nonzero solution to $(E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2})$.

Hence, Equation $(E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2})$ has a nonzero solution if and only if the entry of Table 5.1 is $(-,-)$. We thus obtain the following result:

Lemma 5.3. *Equation* $(E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2})$ *has a nonzero pair of rational solutions if and only if* $A = B = \frac{1}{2}$ *with set of steps* $\mathcal{S}_3$*. In this case, the solution is given by the pair* $(\frac{1}{\widetilde{\gamma}_1}, -\frac{1}{\widetilde{\gamma}_2})$.

	$\mathcal{S}_1$	$\mathcal{S}_2$	$\mathcal{S}_3$	$\mathcal{S}_4$	$\mathcal{S}_5$
$(A,B) = (0,0)$	$(+,+)$				
$(A,B) = (0,\frac{1}{2})$	$(-,+)$	$(\perp,4)$	$(+,-)$	$(\perp,4)$	
$(A,B) = (\frac{1}{2},0)$	$(-,+)$	$(\perp,2)$	$(-,+)$		$(\perp,2)$
$(A,B) = (\frac{1}{2},\frac{1}{2})$	$(+,+)$		$(-,-)$	$(\perp,4)$	
$A+B=1$ and $(A,B) \neq (\frac{1}{2},\frac{1}{2})$			$(\perp,2)$		
A generic[2]	$(\perp,2)$				
B generic[2]	$(\perp,4)$				

Table 5.1. The table summarizing solutions to the homogeneous equation $(E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2})$. Note that all cases are handled, as A and B are never equal to 1. We refer to the proof below for details on the signed entries.

Proof of Table 5.1. First, the entries of type $(\perp, i)$ can be directly checked by looking at the tables in computed in Section 4. It remains to prove the "signed" entries. To do this, we will apply Lemmas 5.1 and 5.2 for some sets of steps and various values of A and B to exhibit the signed solutions to $(E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2})$.

---

[2] A generic means that $A \neq \{0, \frac{1}{2}\}$ and $A + B \neq 1$ and B generic means that $B \neq \{0, \frac{1}{2}\}$.

We first tell how to build the first four lines, that correspond to the cases $(A,B)\in\{0,\frac{1}{2}\}^2$. These conditions on A and B correspond to the fact that each individual function $\widetilde{\gamma}_1$ (depending on A) and $\widetilde{\gamma}_2$ (depending on B) admits a *signed decoupling*. More precisely, assume that we restrict to some set of steps $\mathcal{S}_i$ with $1\le i\le 5$, and some value of $(A,B)\in\{0,\frac{1}{2}\}^2$. If one writes

$$\widetilde{\gamma}_1 = h_{1,1}\cdot h_{1,2} \text{ with } h_{1,1}^{\iota_1}=\varepsilon_{1,1}h_{1,1} \text{ and } h_{1,2}^{\iota_2}=\varepsilon_{1,2}h_{1,2}$$

and

$$\widetilde{\gamma}_2 = h_{2,1}\cdot h_{2,2} \text{ with } h_{2,1}^{\iota_1}=\varepsilon_{2,1}h_{2,1} \text{ and } h_{2,2}^{\iota_2}=\varepsilon_{2,2}h_{2,2},$$

then one obtains the following signed solution to ($E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2}$) for models of set of steps $\mathcal{S}_i$ and (A,B) having the prescribed value:

$$(h_1,h_2)=\left(\frac{h_{2,1}}{h_{1,1}},\frac{h_{1,2}}{h_{2,2}}\right) \text{ with } h_1^{\iota_1}=\varepsilon_{1,1}\varepsilon_{2,1}h_1=\varepsilon_1h_1 \text{ and } h_2^{\iota_2}=\varepsilon_{1,2}\varepsilon_{2,2}h_2=\varepsilon_2h_2.$$

We thus only give below the signed decouplings of $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ relatively to the given set of steps and weights A, B. They cover the first four lines of Table 5.1. We write below $x=x(s)$ and $y=y(s)$.

(1) **Signed decouplings of $\widetilde{\gamma}_1$:**
   (a) **$A=0$, any set of steps:** In this case, we have $\widetilde{\gamma}_1=-td_{-1,1}\frac{1}{y}\in\mathbb{C}(y)$, hence

   $$\widetilde{\gamma}_1=h_{1,2} \text{ with } h_{1,2}^{\iota_2}=h_{1,2}.$$

   (b) **$A=\frac{1}{2}$, sets of steps $\mathcal{S}_1$, $\mathcal{S}_3$ and $\mathcal{S}_4$ (equivalently: $d_{1,0}=0$)**: In this case, it is easy to check that

   $$(x\widetilde{\gamma}_1)^2=\tfrac{1}{4}-t^2d_{1,-1}d_{-1,1}-t^2d_{1,-1}d_{0,1}x-t^2d_{1,-1}d_{1,1}x^2\in\mathbb{C}[x].$$

   This polynomial is never a square in $\mathbb{C}[x]$:
   - When $d_{1,1}=0$, it has degree 1 in the variable x because then $d_{0,1}\neq 0$.
   - Otherwise, it has degree 2 in the variable x, with discriminant Δ equal to

   $$\begin{aligned}\Delta&=t^4d_{1,-1}^2d_{0,1}^2+(1-4t^2d_{1,-1}d_{-1,1})t^2d_{1,-1}d_{1,1}\\&=d_{1,-1}d_{1,1}t^2+(d_{1,-1}^2d_{0,1}^2-4d_{1,-1}d_{-1,1})t^4.\end{aligned}$$

   The coefficient in t^2 is nonzero since $d_{1,-1}$ and $d_{1,1}$ are nonzero. Hence, since t is transcendental over the field $\mathbb{Q}(d_{i,j})\subset\mathbb{F}$, we deduce that $\Delta\neq 0$.

   Therefore, $x\widetilde{\gamma}_1$ does not belong to $\mathbb{C}(x(s))$ while $(x\widetilde{\gamma}_1)^2$ does. Since $\mathbb{C}(s)/\mathbb{C}(x)$ is Galois with Galois group ι_1, these conditions translate into $(x\widetilde{\gamma}_1)^{\iota_1}\neq x\widetilde{\gamma}_1$ and $\left((x\widetilde{\gamma}_1)^2\right)^{\iota_1}=(x\widetilde{\gamma}_1)^2$, so $(x\widetilde{\gamma}_1)^{\iota_1}=-x\widetilde{\gamma}_1$, and

   $$\widetilde{\gamma}_1=h_{1,1} \text{ with } h_{1,1}^{\iota_1}=-h_{1,1}.$$

(2) **Signed decouplings of $\widetilde{\gamma}_2$:**
   (a) **$B=0$, any set of steps**: In this case, we have $\widetilde{\gamma}_2=-td_{-1,1}\frac{1}{x}\in\mathbb{C}(x)$, and thus

   $$\widetilde{\gamma}_2=h_{2,1} \text{ with } h_{2,1}^{\iota_1}=h_{2,1}.$$

   (b) **$B=\frac{1}{2}$, set of steps $\mathcal{S}_1$**: In this case, from (iii) of Lemma 5.1 with $\lambda=\frac{1}{2}$ we have

   $$\mu\overset{\text{def}}{=}-(y\widetilde{\gamma}_2)u_\lambda=-(\tfrac{1}{2}-td_{-1,1}\tfrac{x}{y})u_\lambda=\tfrac{1}{4}-t^2d_{1,-1}d_{-1,1}-\tfrac{1}{2}td_{0,1}y\in\mathbb{C}[y].$$

   Moreover, from (ii) of Lemma 5.1 with $\lambda=\frac{1}{2}$, then

   $$u_\lambda^2=\tfrac{1}{4}-t^2d_{1,-1}d_{-1,1}-t^2d_{1,-1}d_{0,1}x\in\mathbb{C}[x].$$

This polynomial is not a square in $\mathbb{C}(x)$ because it has degree 1 in x, so $u_\lambda^{\iota_1} = -u_\lambda$. Reasoning as above, we deduce

$$\widetilde{\gamma}_2 = h_{2,1}\cdot h_{2,2} \overset{\text{def}}{=} \tfrac{1}{u_\lambda}\cdot\left(-\tfrac{\mu}{y}\right) \text{ with } h_{2,1}^{\iota_1} = -h_{2,1} \text{ and } h_{2,2}^{\iota_2} = h_{2,2}.$$

(c) **$B = \frac{1}{2}$, set of steps $\mathcal{S}_3$**: In this case, from (ii) of Lemma 5.2 we have

$$(y\widetilde{\gamma}_2)^2 = \tfrac{1}{4} - d_{1,-1}d_{-1,1}t^2 - d_{1,1}d_{-1,1}t^2y^2 \in \mathbb{C}[y].$$

This polynomial is not a square in $\mathbb{C}[y]$. Indeed, it has degree 2, and its discriminant Δ is equal to

$$\Delta = (1 - 4d_{1,-1}d_{-1,1}t^2)d_{1,1}d_{-1,1}t^2 = d_{1,1}d_{-1,1}t^2 + O(t^4).$$

As t is transcendental over the field of parameters $\mathbb{Q}(d_{i,j}) \in \mathbb{F}$, then Δ is always nonzero since $d_{1,1}$ and $d_{-1,1}$ are nonzero. Therefore, $(y\widetilde{\gamma}_2)^{\iota_2} = -y\widetilde{\gamma}_2$, from which we deduce

$$\widetilde{\gamma}_2 = h_{2,2} \text{ with } h_{2,2}^{\iota_2} = -h_{2,2}.$$

There now remains to fill line 5 of Table 5.1, which corresponds to the case of $A + B = 1$ for sets of steps $\mathcal{S}_1$ and $\mathcal{S}_2$. In this case, we have from (ii) of Lemma 5.1 with $\lambda = A$ that

$$(x\widetilde{\gamma}_1)u_\lambda = \lambda(1-\lambda) - t^2d_{1,-1}d_{-1,1} - (\lambda t d_{1,0} + t^2 d_{1,-1}d_{0,1})x \in \mathbb{C}[x].$$

Moreover, from (iii) of Lemma 5.1 with $\lambda = A$, then

$$-(y\widetilde{\gamma}_2)u_\lambda = \lambda(1-\lambda) - t^2d_{1,-1}d_{-1,1} - ((1-\lambda)td_{0,1} + t^2d_{-1,1}d_{1,0})y \in \mathbb{C}[y].$$

Note that $u_\lambda \neq 0$, for $(x\widetilde{\gamma}_1)u_\lambda$ is a nonzero polynomial in $\mathbb{C}[x]$ (the constant coefficient is a nonzero polynomial in $\mathbb{F}[t]$, for t is transcendental over $\mathbb{F}$ and $d_{1,-1}d_{-1,1} \neq 0$), x transcendental over $\mathbb{C}$. Therefore, the pair

$$(h_1, h_2) \overset{\text{def}}{=} \left(\frac{x}{x\widetilde{\gamma}_1 u_\lambda}, -\frac{y}{y\widetilde{\gamma}_2 u_\lambda}\right) \text{ with } h_1^{\iota_1} = h_1 \text{ and } h_2^{\iota_2} = h_2$$

is a signed solution to $(E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2})$. ☐

5.2. One particular case: support $\mathcal{S}_1$, $B = \frac{1}{2}$ and $A \neq \frac{1}{2}$.

In the previous subsection, we were able to give a uniform proof for determining which parameters and supports allow for nonzero solutions to $(E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2})$. For $(E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega})$, we were not able to find a uniform argument, for one edge case remains, that we thus treat aside in this section. The remaining cases (i.e. $\mathcal{S}_1$ with $B \neq \frac{1}{2}$ or $A = \frac{1}{2}$ or the other set of steps) are then treated in Section 5.3.

This edge case concerns models with set of steps $\mathcal{S}_1$ ($d_{1,1} = d_{1,0} = 0$) and Boltzmann weights satisfying $B = \frac{1}{2}$ and $A \neq \frac{1}{2}$. We then show that the generating function $Q(x,y)$ is non-D-algebraic in x or y.

Following Lemma 5.1, let

$$u \overset{\text{def}}{=} u_{1/2} = \tfrac{1}{2} - td_{1,-1}\tfrac{x}{y} = -(\tfrac{1}{2} - td_{0,1}y - td_{-1,1}\tfrac{y}{x}) = \frac{1}{2} - A + x\widetilde{\gamma}_1.$$

This function satisfies the following relations.

Lemma 5.4. *Write $x = x(s)$ and $y = y(s)$. The following identities hold:*

(i) $u^2 = \frac{1}{4} - t^2d_{-1,1}d_{1,-1} - t^2d_{1,-1}d_{0,1}x \in \mathbb{C}[x]$ *and* $u^{\iota_1} = -u$,

(ii) $-(y\widetilde{\gamma}_2)u = \frac{1}{4} - \frac{1}{2}td_{0,1}y - t^2d_{-1,1}d_{1,-1} \in \mathbb{C}[y]$.

Proof. The algebraic identities of (i) and (ii) are a direct consequence of Lemma 5.1.

Moreover, $u^2 \in \mathbb{C}(x)$, while from (i) this polynomial is not a square in $\mathbb{C}(x)$ (indeed, it has degree 1 in x). Since $\mathbb{C}(s)/\mathbb{C}(x)$ is Galois with Galois group generated by ι_1, we have $(u^2)^{\iota_1} = u^2$ and $u^{\iota_1} \neq u$, hence $u^{\iota_1} = -u$. □

We investigate the solutions of $(E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega})$.

Lemma 5.5. *If (h_1, h_2) is a solution to $(E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega})$, then $(h_2)_\infty = k \cdot (\iota_2 P_4 + P_4) + p \cdot (0 + \infty)$ for some $p \geq 0$ and $k \in \{0, 1\}$.*

Proof. From Table A.1 and Proposition 3.10, we observe that the only nonnegative entry of the matrix M_2 is $\delta(\iota_2 P_4, P_4) = 1$. Therefore, from Lemma 3.7, we see that if (h_1, h_2) is a pair of solutions to $(E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega})$, then the poles of h_2 must belong to $\{P_4, \iota_2 P_4, 0, \infty\}$.

We now bound the order of the pole P_4 of h_2. We show that if it has order greater than 1, then P_4 is a pole of h_1, and deduce a contradiction. For the first part, we use that h_2 is a solution to $(E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega})$:

$$\widetilde{\gamma}_1 h_1 + \widetilde{\gamma}_2 h_2 + \omega = 0.$$

We first note that P_4 is a zero of order 1 of $\widetilde{\gamma}_2$ (the zeros of $\widetilde{\gamma}_1$ are computed in the Maple worksheet dedicated to the set of steps $\mathcal{S}_1$). Now, assume that P_4 is a pole of h_2 of order greater than 1. As P_4 is a zero of order 1 of $\widetilde{\gamma}_2$, we deduce that P_4 is a pole of $\widetilde{\gamma}_2 h_2$. Then from $(E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega})$, we deduce that P_4 is a pole of $\widetilde{\gamma}_1 h_1$. As $P_4 \neq 0, \infty$, it is not a pole of $\widetilde{\gamma}_1$ (Proposition 1.4), thus P_4 is a pole of h_1.

Since P_4 is a pole of h_1, then by Lemma 3.7, there must exist $Q^+ \in \mathcal{L}^+$ such that $\sigma^n P_4 = Q^+$ for some $n \geq 0$. But since $B = \frac{1}{2}$, Table A.1 implies that $\delta(\iota_2 P_4, P_4) = \delta(\sigma^{-1} P_4, P_4) = 1$. Therefore, we deduce that

$$\delta(\iota_2 P_4, Q^+) = \delta(\iota_2 P_4, P_4) + \delta(P_4, Q^+) = 1 + n \geq 1.$$

This is a contradiction, since no entry of the line corresponding to $\iota_2 P_4$ in Table A.1 is positive.

Therefore, the pole P_4 has order 0 or 1. Since h_2 satisfies $h_2^{\iota_2} = h_2$, the point $\iota_2 P_4$ has the same order as P_4 as a pole of h_2, and Table A.1 and the fact that $B = \frac{1}{2}$ asserts that $P_4 \neq \iota_2 P_4$, hence the result. □

Lemma 5.6. *The function $u\widetilde{\gamma}_2$ has divisor $(u\widetilde{\gamma}_2) = P_4 + \iota_2 P_4 - 0 - \infty$.*

Proof. See the dedicated section in the Maple worksheet covering the set of steps $\mathcal{S}_1$. □

We can now state the classification for this support and parameters.

Proposition 5.7. *For every weighted model of the set of steps $\mathcal{S}_1$, if $B = \frac{1}{2}$ and $A \neq \frac{1}{2}$, then the series $Q(x, y)$ is non-D-algebraic in x and y.*

Proof. Assume that $Q(x, y)$ is x-D-algebraic or y-D-algebraic. By Proposition 1.6, Theorem 2.1 and Lemma 2.2, then $\widetilde{F}(s) = \widetilde{f}(s) + \widetilde{f}_h(s)$, $\widetilde{G}(s) = \widetilde{g}(s) + \widetilde{g}_h(s)$, with $(\widetilde{f}, \widetilde{g})$ solution to $(E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega})$ and $(\widetilde{f}_h, \widetilde{g}_h)$ solution to $(E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2})$. As $(E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2})$ has no nonzero rational solution by Lemma 5.3, this implies that $\widetilde{f}_h = \widetilde{g}_h = 0$, thus $\widetilde{F}(s) = \widetilde{f}(s)$ and $\widetilde{G}(s) = \widetilde{g}(s)$. We now distinguish between two cases, depending on the value of k in Lemma 5.5 ($k = 0$ or $k = 1$).

- If $k = 0$, then $(\widetilde{g})_\infty = p \cdot (0 + \infty)$. Thus, by Lemma 2.4, we have that $\widetilde{g}(s) = H(1/y(s))$ for $H(y) \in \mathbb{F}[y]$ a polynomial. Thus, Proposition 1.7 implies that $yQ(0, y) = H(1/y)$, which is absurd since $Q(0, y) = 1 + O(y)$.

- Otherwise, $k = 1$, and $(\widetilde{g})_\infty = P_4 + \iota_2 P_4 + p\cdot(0+\infty)$. We thus have from Lemma 5.6

  $$(\widetilde{g}\cdot(u\widetilde{\gamma}_2)) = (\widetilde{g})_0 + P_4 + \iota_2 P_4 - P_4 - \iota_2 P_4 - (p+1)\cdot(0+\infty) = (\widetilde{g})_0 - (p+1)\cdot(0+\infty).$$

  Hence, the poles of $\widetilde{g}\cdot(u\widetilde{\gamma}_2)$ belong to $\{0,\infty\}$. Furthermore, we have from (ii) of Lemma 5.4 that $-4u\widetilde{\gamma}_2 = \frac{1-2td_{0,1}y(s)-4t^2d_{-1,1}d_{1,-1}}{y(s)} \in \mathbb{C}(y(s))$, thus $\widetilde{g}\cdot(u\widetilde{\gamma}_2)$ is fixed by ι_2. Thus, Lemma 2.4 implies that $\widetilde{g}\cdot(-4u\widetilde{\gamma}_2) = H(1/y(s))$ for some polynomial $H(y)\in\mathbb{F}[y]$, and thus

  $$\widetilde{g}(s) = \frac{y(s)}{1-2td_{0,1}y(s)-4t^2d_{-1,1}d_{1,-1}}H(1/y(s))$$

  for some polynomial $H(y)\in\mathbb{F}[y]$. Thus, Proposition 1.7 implies that

  $$yQ(0,y) = \frac{y}{1-2td_{0,1}y-4t^2d_{-1,1}d_{1,-1}}H(1/y).$$

  Since

  $$yQ(0,y) = y + O(y^2)$$

  and

  $$\frac{y}{1-2td_{0,1}y-4t^2d_{-1,1}d_{1,-1}}H(1/y) = \frac{H(0)}{1-4t^2d_{-1,1}d_{1,-1}} + O(1/y),$$

  this implies that $H(y) = \mu$ a constant in $\mathbb{F}$, and thus that $\widetilde{g}(s) = \mu/u$ for some $\mu\in\mathbb{F}$.

  Therefore, we may rewrite $(E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega})$ as

  $$\widetilde{\gamma}_1\widetilde{f} - \tfrac{\mu}{u} + \omega = 0. \tag{5.1}$$

  But $u^{\iota_1} = -u$ by (i) of Lemma 5.4, thus $(x\widetilde{\gamma}_1)^{\iota_1} = (A-\frac{1}{2}+u)^{\iota_1} = A-\frac{1}{2}-u$, thus $(\widetilde{\gamma}_1)^{\iota_1} + (\widetilde{\gamma}_1) = (2A-1)/x$. Moreover, $\omega = 1-A-B = \frac{1}{2}-A$ since $B=\frac{1}{2}$. Thus, by taking $\iota_1(5.1)+(5.1)$, one obtains the identity

  $$(2A-1)\,\tfrac{\widetilde{f}}{x} - (2A-1) = 0.$$

  As $A\neq\frac{1}{2}$, this implies that $\widetilde{f}(s) = x(s)$. By Proposition 1.7, this implies that $Q(x,0)=1$, which is absurd since $a,b>0$. ☐

5.3. Full classification.

We now state and prove the full classification:

Theorem 5.8. *For any weighted genus 0 model, the generating function $Q(x,y)$ of weighted walks in the quadrant with interacting boundaries has the following nature in the variables x and y:*

1. *For all models of set of steps $\mathcal{S}_1$ or $\mathcal{S}_2$, and if $a+b=ab$, the generating function $Q(x,y)$ is* ***rational*** *with partial series $Q(x,0)$ and $Q(0,y)$ respectively equal to*

   $$Q(x,0) = \frac{1}{1-x\dfrac{ad_{1,0}t+abd_{1,-1}d_{0,1}t^2}{1-abd_{1,-1}d_{-1,1}t^2}}, \qquad Q(0,y) = \frac{1}{1-y\dfrac{bd_{0,1}t+abd_{-1,1}d_{1,0}t^2}{1-abd_{1,-1}d_{-1,1}t^2}}.$$

2. *For all models of set of steps $\mathcal{S}_3$ where $a=b=2$, the generating function $Q(x,y)$ is* ***algebraic*** *of degree 4, with partial series $Q(x,0)$ and $Q(0,y)$ respectively equal to*

   $$Q(x,0) = \frac{1}{\sqrt{1-x^2\dfrac{4d_{1,1}d_{1,-1}t^2}{1-4d_{1,-1}d_{-1,1}t^2}}}, \qquad Q(0,y) = \frac{1}{\sqrt{1-y^2\dfrac{4d_{1,1}d_{-1,1}t^2}{1-4d_{1,-1}d_{-1,1}t^2}}}.$$

(3) In all other cases, the series $Q(x,y)$ is ***neither x-D-algebraic nor y-D-algebraic****.*

Proof. We prove all the points in order. We begin with 1. Recall that as $A = 1-\frac{1}{a}$ and $B = 1-\frac{1}{b}$, we have $a+b = ab$ is equivalent to $A+B = 1$, which implies that $\omega = 0$. From (ii) and (iii) of Lemma 5.1 with $\lambda = A$, we see that the pair $\left(\frac{x(s)}{x(s)\widetilde{\gamma}_1 u_A}, -\frac{y(s)}{y(s)\widetilde{\gamma}_2 u_A}\right) \in \mathbb{C}(x(s)) \times \mathbb{C}(y(s))$ is a nonzero solution to $(E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega})$. Therefore, by (1) of Lemma 2.7 and (i) and (ii) of Lemma 5.1, there exists $\lambda \in \mathbb{C}$ such that

$$Q(x,0) = \frac{\lambda}{AB - t^2 d_{1,-1}d_{0,1}x - t^2 d_{1,-1}d_{-1,1} - Atd_{1,0}x}$$

$$\text{and } Q(0,y) = \frac{\lambda}{AB - t^2 d_{1,-1}d_{1,0}y - t^2 d_{1,-1}d_{-1,1} - Btd_{0,1}y}.$$

We know that $Q(0,0) = 1$, hence by substituting $x = 0$ in $Q(x,0)$ (or $y = 0$ in $Q(0,y)$), we find $\lambda = AB - t^2 d_{1,-1}d_{-1,1}$. We obtain the identities claimed in the theorem using $\frac{1}{A} = b$ and $\frac{1}{B} = a$ (this uses $a+b = ab$).

We now prove 2. In this case, we have from Lemma 5.3 that $(E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2})$ admits a nonzero solution $\left(\frac{1}{\widetilde{\gamma}_1}, -\frac{1}{\widetilde{\gamma}_2}\right)$. Therefore, by (2) of Lemma 2.7 and Lemma 5.2, there exists $\lambda \in \mathbb{C}$ such that

$$Q(x,0) = \frac{\lambda}{\sqrt{\frac{1}{4} - d_{1,-1}d_{-1,1}t^2 - d_{1,1}d_{-1,1}x^2t^2}} \quad \text{and } Q(0,y) \quad = \frac{\lambda}{\sqrt{\frac{1}{4} - d_{1,-1}d_{-1,1}t^2 - d_{1,1}d_{-1,1}y^2t^2}}.$$

We know that $Q(0,0) = 1$, thus $\lambda = \sqrt{\frac{1}{4} - d_{1,-1}d_{-1,1}t^2}$, and we get the expression in the theorem.

Now, it remains to prove 3, namely that for all other cases $Q(x,y)$ is non-D-algebraic in x and y. Depending on the case, we use one of the three arguments below.

(1) If we are in the situation of Section 5.2, then $Q(x,y)$ is non-D-algebraic in x and y by Proposition 5.7.

(2) If the entries of M_1 satisfy $(M_1)_{i,j} \in \mathbb{Z}^- \cup \{\perp\}$, then by Lemma 3.11, if (h_1,h_2) is a rational solution to $(E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega})$, the poles of h_1 must belong to $\{0,\infty\}$. Moreover, Lemma 5.3 asserts that there is no nonzero rational solution to $(E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2})$ (the only situation where it happens corresponds to point 2 of the present theorem, and has already been covered). Therefore, by Lemma 2.5, the generating function $Q(x,y)$ is non-D-algebraic in x and y.

(2') Likewise, if the entries of M_2 satisfy $(M_2)_{i,j} \in \mathbb{Z}^- \cup \{\perp\}$, then for (h_1,h_2) a solution to $(E_{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\omega})$ the poles of h_2 must belong to $\{0,\infty\}$ by Lemma 3.11. Moreover, Lemma 5.3 asserts that there is no nonzero rational solution to $(E'_{\widetilde{\gamma}_1,\widetilde{\gamma}_2})$. Therefore, by Lemma 2.5, the generating function $Q(x,y)$ is non-D-algebraic in x and y.

We check that for any value of the parameters, we are in one of the three above cases (see Section 4).

	Support 1	Support 2	Support 3	Support 4	Support 5
(1) Proposition 5.7	$A+B \neq 1 \wedge B = \frac{1}{2}$				
(2) $(M_1)_{i,j} \in \mathbb{Z}^- \cup \{\perp\}$		$A+B \neq 1$	$A \neq \frac{1}{2}$		
(2') $(M_2)_{i,j} \in \mathbb{Z}^- \cup \{\perp\}$	$A+B \neq 1 \wedge B \neq \frac{1}{2}$		$B \neq \frac{1}{2}$	always	
Algebraic solution	$A+B = 1$		$A = B = \frac{1}{2}$		

The above table gives the exact argument for each value of the parameters, and one can check that no case is missing. ☐

6. Conclusion and comments

In Theorem 5.8, we showed how the addition of the Boltzmann weights affects the nature of the generating function $Q(x, y)$ of walks with interacting boundaries for weighted models of genus 0. Namely, for the first two sets of steps, the relation $a + b = ab$ between the weights makes the series $Q(x, y)$ rational; for the third set of steps the relation $a = b = 2$ makes the series $Q(x, y)$ algebraic; while other Boltzmann weights and other sets of steps keep the series non x-D-finite nor y-D-finite. We now give some perspectives based on these results.

6.1. **Phase transitions.** Regarding the sets of steps $\mathcal{S}_1$ and $\mathcal{S}_2$, one may note that since there is an infinite number of Boltzmann weights a, b such that $Q(x, y)$ is explicit, the question of phase transitions introduced in [TOR14] can be partially treated on the curve $a + b = ab$ (a hyperbola).

Recall that the phases are defined as follows. Let $\mathcal{S}$ be a weighted model. For $n \geq 0$, denote $\mathbb{P}_n$ the probability on the walks using n steps defined by

$$\mathbb{P}_n(w) = \frac{\left(\prod_{(i,j)\in\mathcal{S}} d_{i,j}^{n_{i,j}}\right) a^{n_x} b^{n_y}}{[t^n]Q(1,1)}$$

(i.e. the probability of a walk using n steps is proportional to the numerator in the above equation). Define

$$\mathcal{A} \overset{\text{def}}{=} \limsup_n \mathbb{P}_n(\{w \text{ walk of } n \text{ steps} : w \text{ terminates on the } x\text{-axis}\})$$

and

$$\mathcal{B} \overset{\text{def}}{=} \limsup_n \mathbb{P}_n(\{w \text{ walk of } n \text{ steps} : w \text{ terminates on the } y\text{-axis}\}).$$

These limits correspond respectively to $\mathcal{A}$ and $\mathcal{C}$ in [TOR14]). Four phases are then defined as follows:

1. if $\mathcal{A} = \mathcal{B} = 0$, then the phase is *free* (the walk moves away from the axes),
2. if $\mathcal{A} > 0$ and $\mathcal{B} = 0$, then the phase is *x-attracted* (the walk moves away from the y-axis, and tends to come back infinitely often on the x-axis),
3. if $\mathcal{A} = 0$ and $\mathcal{B} > 0$, then the phase is *y-attracted* (the walk moves away from the x-axis, and tends to come back infinitely often on the y-axis),
4. if $\mathcal{A} > 0$ and $\mathcal{B} > 0$, then the phase is *supercritical* (the walk tends to come in contact with both axes infinitely often).

The values $\mathcal{A}$ and $\mathcal{B}$ can be expressed using the generating function $Q(x, y)$ as

$$\mathcal{A} = \limsup_n \frac{[t^n]Q(1,0)}{[t^n]Q(1,1)} \qquad \mathcal{B} = \limsup_n \frac{[t^n]Q(0,1)}{[t^n]Q(1,1)}.$$

In the case of $\mathcal{S}_1$ or $\mathcal{S}_2$, the function $Q(x, y)$ is rational, hence the singularity analysis on the poles is straightforward. For instance, for the model $\mathcal{S}_2$ with $d_{i,j} = 1$, it yields the phase diagram in Figure 6.1a below (see the Maple worksheet).

The change of nature of the generating function $Q(x, y)$ on this curve, which contains the critical point (a_0, b_0) at the junction of the four phases suggests that this curve could be related to the phase transitions of the walk. Numerical computations based on the first coefficients of $Q(x, y)$ allow us to conjecture that the full phase diagram looks like Figure 6.1b.

(A) Phase diagram on $a+b=ab$. (B) Conjectured phase diagram.

FIGURE 6.1. Phase diagrams for model $\mathcal{S}_2$ with $d_{-1,1}=d_{1,-1}=d_{1,0}=d_{0,1}=1$.

It would be interesting to know whether the knowledge of the phases on the curve $a+b=ab$ is enough to deduce some parts of the phase diagram 6.1b, mainly the part under the curve.

6.2. **Combinatorial interpretation.** For models $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$, we found $\mathbb{N}$-algebraic solutions for $Q(x,y)$ when the weights are subject to some relations (i.e. $a+b=ab$ for $\mathcal{S}_1$ and $\mathcal{S}_2$; $a=b=2$ for $\mathcal{S}_3$). These relations were found indirectly through the study of the q-difference equation. These relations being simple enough, one may wonder if the expressions found in (1) and (2) of Theorem 5.8 for those weights may be deduced through a more combinatorial argument.

Regarding the weights $a=b=2$ for $\mathcal{S}_3$, Andrew Elvey-Price pointed out in a private communication with the author a direct proof through an adaptation of the reflection principle. The generating function of unconstrained two dimensional walks using the set of steps $\mathcal{S}_3$ that terminate on the x-axis is easily found to be

$$F(x,t)=\frac{1}{\sqrt{1-4xtd_{1,-1}\left(xtd_{1,1}+\frac{1}{x}td_{-1,1}\right)}}.$$

A clever adaptation of the reflection principle allows us to relate walks with interacting boundaries with Boltzmann weights $a=b=2$ to these walks, through the following identity

$$Q(x,0)\cdot\frac{1}{\sqrt{1-4t^2d_{1,-1}d_{-1,1}}}=F(x,t),$$

which allows us to deduce the form of $Q(x,0)$. We detail this argument and extend it to other sets of steps in an upcoming paper.

Such a direct proof is yet to be found regarding the weights $a+b=ab$, and would be enlightening. For instance, it could give an alternative explanation as why the relation $a+b=ab$ changes the nature of $Q(x,y)$, and hopefully permit to find other sets of steps for which this relation between the Boltzmann weights yield a D-algebraic generating function.

6.3. **Other q-difference equations.** The general study of rational solutions to q-difference equations has already been investigated before. Depending on different constraints on the coefficients and the relation of the complex number q with regards to these coefficients, there may or may not be a general algorithm to decide whether such solutions exist.

In the general case where the coefficients of the equation and q may share algebraic relations, the problem is undecidable [Abr10].

In a more specific case, when the coefficients depend on one parameter, [AR13] gives an algorithm to determine numerical values of this parameter so that the q-difference equation has a nontrivial rational solution. Since we work with more parameters, and we want to find all the algebraic relations between them so that the equation has solutions, none of these algorithms can be applied verbatim. This justifies the approach taken in this paper.

The author thinks that the approach taken in Section 3, specifically the structure given by Lemma 3.7, might adapt quite easily to the study of other decoupling equations of mixed type (multiplicative and additive). Moreover, we note that our approach works for a general infinite group of the walk, even if $\iota_2\iota_1$ is not presented as a multiplication by q on $\mathbb{P}^1$. More precisely, for a general decoupling equation of the form

$$uh_1 + vh_2 + w = 0$$

for functions u, v, w, $h_1^{\iota_1} = \pm h_1$ and $h_2^{\iota_2} = \pm h_2$ on some curve $\mathcal{C} \subset \mathbb{P}^1 \times \mathbb{P}^1$, the same technique yields similar finite sets $\mathcal{L}_1^-$, $\mathcal{L}_1^+$, $\mathcal{L}_2^-$ and $\mathcal{L}_2^+$, which may be exploited in the same way as in Section 3.

6.4. Extension to models of genus 1.

For the time being, we only performed the systematic classification for walks with interacting boundaries of small steps of genus zero. It would seem natural to extend the methods for the models with small steps of genus 1 of [HS08; DR19; KR12] for the same purpose.

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APPENDIX A. DATA

This appendix shows the result of the computation of the matrix M_1 as defined in (3.2) for each set of step $\mathcal{S}_i$. For clarity, the genus 0 models are subdivided according to their support, and there are thus five different tables. For each table, the value of every entry depends algebraically on the complex parameters $A = 1 - \frac{1}{a}$, $B = 1 - \frac{1}{b}$ and $d_{i,j}$. They are computed for each model in its dedicated Maple worksheet.

Note that from Proposition 3.10, the matrix M_1 is symmetric so only the entries on the upper diagonal are specified, and a simple computation allows us to deduce the entries of M_2 from those of M_1. Note that for each table, the zeros of $\widetilde{\gamma}_1$ that are P_1, P_2, and the zeros of $\widetilde{\gamma}_2$ that are P_3, P_4, are chosen arbitrarily and fixed once and for all for each model.

	$\iota_1 P_1$	$\iota_1 P_2$	$\sigma^{-1} P_3$	$\sigma^{-1} P_4$
P_1	0	-1 if $A = 0$ $\perp$ otherwise	-1	-1 if $B = 0$ $\perp$ otherwise
P_2		0 if $A = \frac{1}{2}$ -2 if $A = 0$ $\perp$ otherwise	-2 if $A = 0$ $\perp$ otherwise	0 if $A + B = 1$ -2 if $A = B = 0$ $\perp$ otherwise
$\iota_2 P_3$			-2	-2 if $B = 0$ $\perp$ otherwise
$\iota_2 P_4$				0 if $B = 0$ -2 if $B = 0$ $\perp$ otherwise

Table A.1. Set of steps $\mathcal{S}_1$

	$\iota_1 P_1$	$\iota_1 P_2$	$\sigma^{-1} P_3$	$\sigma^{-1} P_4$
P_1	$\perp$	-1 if $A = 0$ $\perp$ otherwise	-1	$\perp$
P_2		$\perp$	$\perp$	0 if $A + B = 1$ -2 if $A = B = 0$ $\perp$ otherwise
$\iota_2 P_3$			$\perp$	-2 if $B = 0$ $\perp$ otherwise
$\iota_2 P_4$				$\perp$

Table A.2. Set of steps $\mathcal{S}_2$

	$\iota_1 P_1$	$\iota_1 P_2$	$\sigma^{-1} P_3$	$\sigma^{-1} P_4$
P_1	0 if $A = \frac{1}{2}$ $\perp$ otherwise	-1 if $A = 0$ $\perp$ otherwise	$\perp$	$\perp$
P_2		0 if $A = \frac{1}{2}$ $\perp$ otherwise	$\perp$	$\perp$
$\iota_2 P_3$			-1 if $B = \frac{1}{2}$ $\perp$ otherwise	-2 if $B = 0$ $\perp$ otherwise
$\iota_2 P_4$				-1 if $B = \frac{1}{2}$ $\perp$ otherwise

TABLE A.3. Set of steps $\mathcal{S}_3$

	$\iota_1 P_1$	$\iota_1 P_2$	$\sigma^{-1} P_3$	$\sigma^{-1} P_4$
P_1	0 if $A = \frac{1}{2}$ -1 if (C_4) $\perp$ otherwise	-1 if $A = 0$ $\perp$ otherwise	$\perp$	$\perp$
P_2		0 if $A = \frac{1}{2}$ -1 if (C_4) $\perp$ otherwise	$\perp$	$\perp$
$\iota_2 P_3$			$\perp$	-2 if $B = 0$ $\perp$ otherwise
$\iota_2 P_4$				$\perp$

where $(C_4) \equiv (A = 0) \wedge (4 d_{1,-1} d_{1,1} = {d_{0,1}}^2)$.

TABLE A.4. Set of steps $\mathcal{S}_4$

	$\iota_1 P_1$	$\iota_1 P_2$	$\sigma^{-1} P_3$	$\sigma^{-1} P_4$
P_1	-1 if (C_5) $\perp$ otherwise	-1 if $A = 0$ $\perp$ otherwise	$\perp$	$\perp$
P_2		-1 if (C_5) $\perp$ otherwise	$\perp$	$\perp$
$\iota_2 P_3$			-2 if (C_5') $\perp$ otherwise	-2 if $B = 0$ $\perp$ otherwise
$\iota_2 P_4$				-2 if (C_5') $\perp$ otherwise

where $(C_5) \equiv (A = 0) \wedge (4d_{1,1}d_{-1,1} = {d_{0,1}}^2)$ and $(C_5') \equiv (B = 0) \wedge (4d_{1,1}d_{-1,1} = {d_{1,0}}^2)$.

TABLE A.5. Set of steps $\mathcal{S}_5$

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